Model created in COMSOL Multiphysics 6.4

Transverse Modes for a Symmetric Laser Cavity

Introduction

This application demonstrates how to solve for the eigenfrequencies for the lowest order modes in a symmetric laser cavity using the bidirectional Electromagnetic Waves, Beam Envelopes interface. It is assumed that the cavity is two-dimensional and that there are two identical cylindrical mirrors surrounding the air-filled cavity.

The electric field of an electromagnetic wave propagation in 2D space can be expressed as the sum of a number of Hermite–Gaussian modes

$$E_{zn} = A_n\sqrt{\frac{w_0}{w(x)}}H_n\left(\sqrt{2}\frac{y}{w(x)}\right)\exp\left(-\frac{y^2}{w^2(x)}\right)$$
$$\exp\left(-j\left(k_0x-\left(\frac{1}{2}+n\right)\eta(x)+\frac{k_0y^2}{2R(x)}\right)\right) \tag{1}$$

where propagation is assumed along the positive x direction with the x coordinate measured from the beam waist location, y is the transverse direction with the y coordinate defining the distance from the optical axis, and the field is polarized in the z direction. The minimum spot radius is given by w_0, whereas the spot radius as a function of distance from the beam waist location is given by

$$w(x) = w_0\sqrt{1+\frac{x^2}{z_0^2}}$$

where the Rayleigh range is defined by

$$z_0 = \frac{\pi w_0^2}{\lambda} \tag{2}$$

and λ is the wavelength. H_n is the Hermite polynomial of degree n. The first polynomials are

$$H_0 = 1$$

$$H_1 = 2x$$

and

$$H_2 = 4x^2-2$$

The vacuum wave number is given by

$$k_0 = \frac{2\pi}{\lambda} = \frac{2\pi f}{c} \tag{3}$$

where f is the frequency and c is the speed of light.

Notice that the Gouy phase shift around the beam waist location is given

$$\left(\frac{1}{2} + n\right)\eta(x) \tag{4}$$

where

$$\eta(x) = \operatorname{atan}\frac{x}{z_0} \tag{5}$$

The wavefront curvature is expressed by

$$R(x) = x\left(1 + \frac{z_0^2}{x^2}\right) \tag{6}$$

Notice that in 3D geometry, the square root expression in the amplitude

$$\sqrt{\frac{w_0}{w(x)}}$$

would be replaced by

$$\frac{w_0}{w(x)}$$

and the Gouy phase shift, Equation 4, would be replaced by

$$(1 + m + n)\eta(x)$$

where m is the order for another Cartesian transverse Hermite–Gaussian mode that would depend on the transverse z coordinate.

To match the mode to the cavity, the wavefront curvature of the field at the mirrors must match the mirror curvature, Ref. 1. Thus, for a symmetric cavity, where both mirrors have the curvature R and the cavity length is d, Equation 6 states that

$$R = \frac{d}{2}\left(1 + \frac{4z_0^2}{d^2}\right) \tag{7}$$

Introducing the stability parameter g, defined by

$$g = 1 - \frac{d}{R}$$

Equation 7 requires the Rayleigh range to be

$$z_0 = \sqrt{\frac{1+g}{1-g}}\frac{d}{2} \tag{8}$$

Since the Rayleigh range (see Equation 2) depends on the spot size w_0, it is clear that the cavity spot size depends on the ratio between the mirror curvature and the cavity length. For example, when R is infinite a planar cavity having flat mirrors is considered, $g = 1$, and the Rayleigh range and the spot size become infinitely large. On the other hand, for a concentric cavity, for which $g = -1$, the beam waist spot size is diffraction limited but the spot radius at the mirrors is large. The confocal cavity, for which $g = 0$, gives the smallest possible mode volume for a cavity of a given length.

In this application a cavity having a mirror radius of curvature that is 50% longer than the cavity length is studied.

At resonance, the phase shift for propagation half a cavity round trip must equal a multiple times π. Including also a π phase shift from the reflection at the perfectly conducting mirrors, the phase requirement becomes

$$k_0 d - 2\left(\frac{1}{2} + n\right)\eta\left(\frac{d}{2}\right) = (q+1)\pi$$

where q is the longitudinal mode number (the number of nodes in the axial standing-wave pattern).

Using Equation 3, Equation 5, and Equation 8, the resonance frequency can be expressed as

$$f_{qn} = \frac{c}{2d}\left[q + 1 + \frac{1}{\pi}(1+2n)\operatorname{atan}\sqrt{\frac{1-g}{1+g}}\right] \tag{9}$$

Equation 9 will be used when comparing to the computed resonance frequencies.

Model Definition

In this application, it is assumed that the cavity boundaries are perfectly conducting. That is, the electric field component in the plane of the boundaries is zero. When using the bidirectional formulation for the Electromagnetic Waves, Beam Envelopes interface, this means that the field at the boundaries are defined by

$$E = E_1 \exp(-j\phi) + E_2 \exp(j\phi) = 0$$

where E_1 and E_2 are the solved for slowly varying field envelopes for the forward- and backward-propagating waves, respectively, and ϕ is a rapidly varying predefined phase function. The phase function ϕ depends on the frequency, through the wave number (see Equation 3). If an eigenfrequency study would be used for solving the cavity problem, the frequencies inserted in ϕ would not be the self-consistently solved for eigenfrequencies, but rather the prescribed linearization point frequency f_0. The frequency f_0 will not approximate the eigenfrequencies good enough, so the returned eigenfrequencies will be wrong. Thus, to accurately find the eigenfrequencies, the problem is formulated as a stationary study solving for both the electric field and the frequencies that self-consistently fulfills the resonance condition. The eigenfrequencies are assumed to be real, as the cavity is closed. The equation solved, in addition to the Helmholtz equation, is only an equation for normalizing the electric field.

To solve for the electric field and the resonance frequency, COMSOL's complex splitting functionality is used. Thereby the complex variables are split into their corresponding real and imaginary parts before solving. Then all equations will be real and analytical, so an accurate Jacobian can be calculated. This approach solves the problem faster and more robustly. After solving, the complex variables are formed again from the solved for corresponding real and imaginary parts.

Results and Discussion

Figure 1 shows the electric field norm for the wave propagating from left to right for a mode with transverse mode number n equal to 0. Notice that since the cavity is 1 m long and the wavelength is 1 µm, the longitudinal mode number is close to 2,000,000.

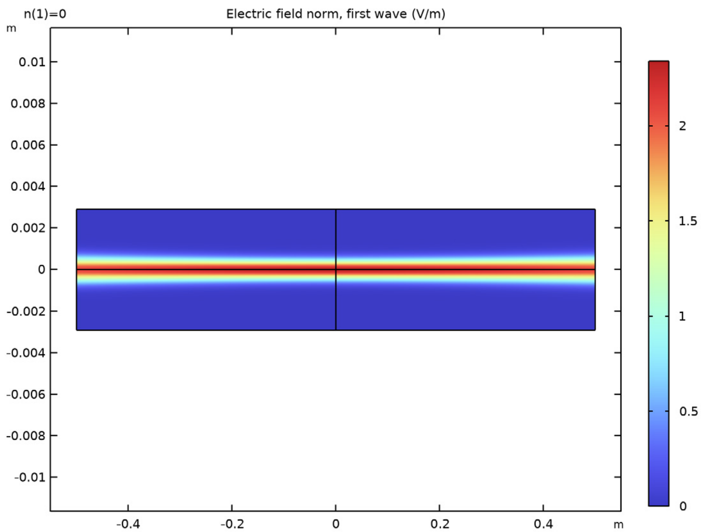


Figure 1: A fundamental transverse mode ($n = 0$).

[Figure 2](#) shows a higher-order mode with transverse mode number equal to 1.

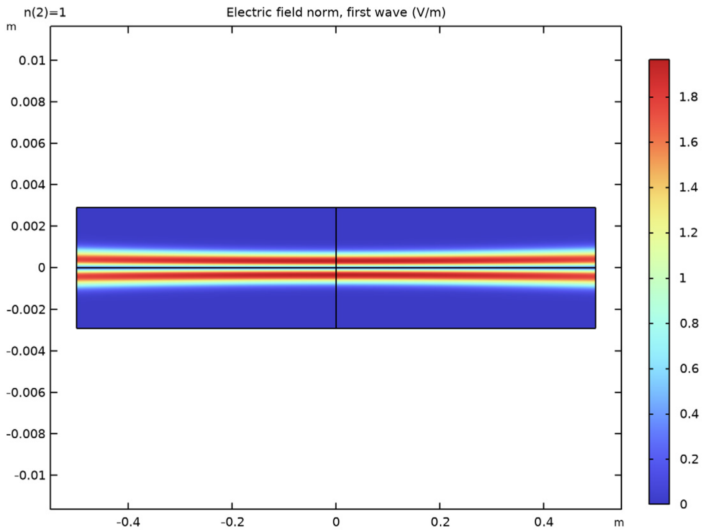


Figure 2: A higher-order mode with transverse mode number 1.

Finally, Figure 3 shows a mode for a transverse mode number equal to 2.

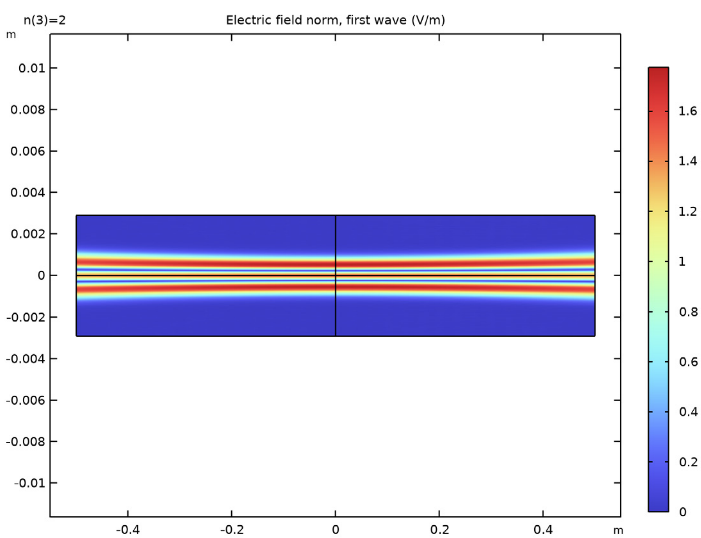


Figure 3: A higher-order mode with transverse mode number 2.

Table 1 shows the analytical resonance frequencies from Equation 9 for the three modes solved for.

TABLE 1: RESONANCE FREQUENCIES FOR LONGITUDINAL MODE NUMBER 2,000,000.

TRANSVERSE MODE NUMBER	RESONANCE FREQUENCY
0	2.9979263726603625E14
1	2.9979269599098925E14
2	2.9979275471086475E14

Reference

1. A. Yariv, *Optical Electronics in Modern Communications*, 5th ed., Oxford University Press, New York, 1997.

Application Library path: `Wave_Optics_Module/Verification_Examples/symmetric_laser_cavity`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **2D**.

2 In the **Select Physics** tree, select **Optics** > **Wave Optics** > **Electromagnetic Waves, Beam Envelopes (ewbe)**.

3 Click **Add**.

4 Click **Study**.

5 In the **Select Study** tree, select **Empty Study**.

6 Click **Done**.

STUDY 1

Step 1: Stationary

In the **Study** toolbar, click **Stationary**.

GLOBAL DEFINITIONS

Parameters 1

Add some parameters that define the geometry and the eigenfrequencies searched for.

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

Name	Expression	Value	Description
wlref	1[um]	1E-6 m	Reference wavelength
d	1[m]	1 m	Cavity length
R	1.5*d	1.5 m	Mirror radius
g	1-d/R	0.33333	Stability parameter
q	floor(2*d/wlref)	2E6	Longitudinal mode number
n	0	0	Transverse mode number
fqn	c_const/(2*d)*(q+1+1/pi*(1+2*n)*atan(sqrt((1-g)/(1+g))))	2.9979E14 rad/s	Mode frequency
wlqn	c_const/fqn	1E-6 m	Mode wavelength
w0	sqrt(d*wlqn/(2*pi)*sqrt((1+g)/(1-g)))	4.7442E-4 m	Spot radius
z0	pi*w0^2/wlqn	0.70711 m	Rayleigh range
wR	sqrt(d*wlqn/pi*sqrt(1/(1-g^2)))	5.8105E-4 m	Spot radius at the mirrors
h0	10*wR	0.0058105 m	Cavity height
k0	2*pi*fqn/c_const	6.2832E6 rad/m	Wave number

Analytic 1 (an1)

1 In the **Home** toolbar, click f(X) **Functions** and choose **Global** > **Analytic**.

2 In the **Settings** window for **Analytic**, type `hermite` in the **Function name** text field.

3 Locate the **Definition** section. In the **Expression** text field, type `if(n==0,1,if(n==1,2*x,if(n==2,4*x^2-2,0)))`.

4 In the **Arguments** text field, type `n,x`.

GEOMETRY 1

The geometry consists of an intersection of two circles, representing the cylindrical mirrors, and a rectangle, defining the cavity length and height. However, since the cavity is symmetric, only one circle and a rectangle is required to build the cavity.

Circle 1 (c1)

1 In the **Geometry** toolbar, click **Circle**.

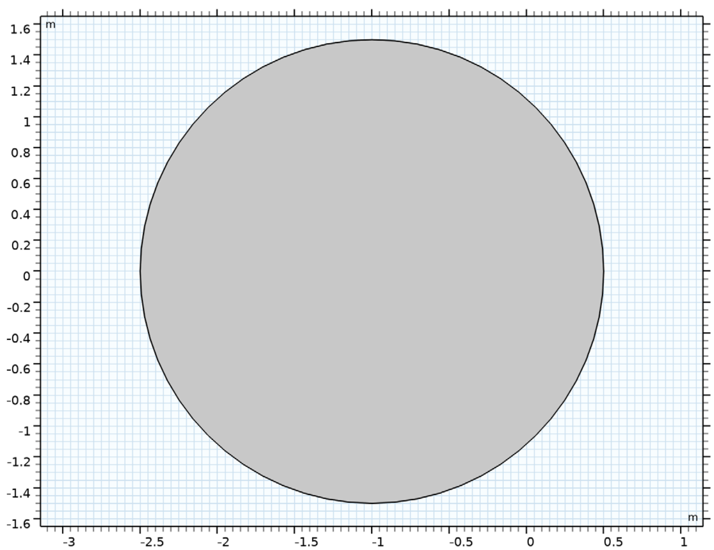
2 In the **Settings** window for **Circle**, locate the **Size and Shape** section.

3 In the **Radius** text field, type `R`.

4 Locate the **Position** section. In the **x** text field, type `d/2-R`.

5 Click **Build Selected**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.



Rectangle 1 (r1)

1 In the **Geometry** toolbar, click **Rectangle**.

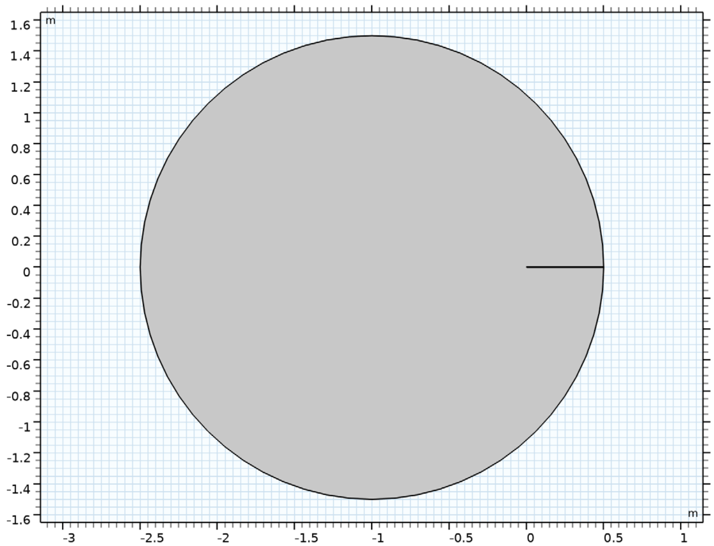
2 In the **Settings** window for **Rectangle**, locate the **Size and Shape** section.

3 In the **Width** text field, type `d/2`.

4 In the **Height** text field, type `h0/2`.

5 Click **Build Selected**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.



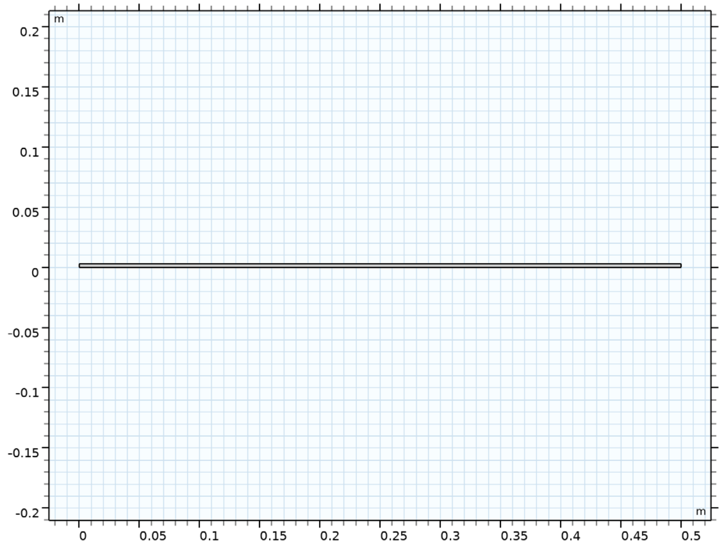
Intersection 1 (int1)

1 In the **Geometry** toolbar, click **Booleans and Partitions** and choose **Intersection**.

2 Click in the **Graphics** window and then press Ctrl+A to select both objects.

3 In the **Settings** window for **Intersection**, click **Build Selected**.

4 Click the **Zoom Extents** button in the **Graphics** toolbar.



Mirror 1 (mir1)

1 In the **Geometry** toolbar, click **Transforms** and choose **Mirror**.

2 Select the object **int1** only.

3 In the **Settings** window for **Mirror**, locate the **Input** section.

4 Select the **Keep input objects** checkbox.

5 Click **Build Selected**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.

Mirror 2 (mir2)

1 In the **Geometry** toolbar, click **Transforms** and choose **Mirror**.

2 Click in the **Graphics** window and then press Ctrl+A to select both objects.

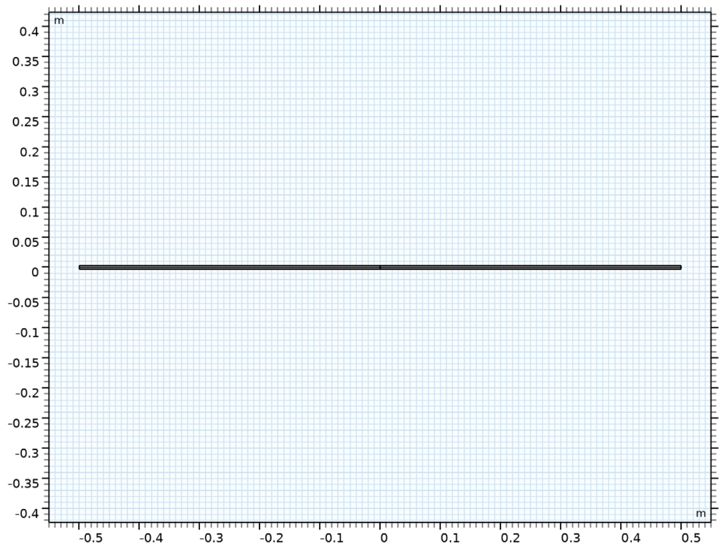
3 In the **Settings** window for **Mirror**, locate the **Normal Vector to Line of Reflection** section.

4 In the **x** text field, type 0.

5 In the **y** text field, type 1.

6 Locate the **Input** section. Select the **Keep input objects** checkbox.

7 Click **Build All Objects**.



DEFINITIONS

Create another view that make the cavity better fill the graphics window.

View 2

In the **Model Builder** window, under **Component 1 (comp1)** right-click **Definitions** and choose **View**.

Axis

1. In the **Model Builder** window, expand the **View 2** node, then click **Axis**.
2. In the **Settings** window for **Axis**, locate the **Axis** section.
3. From the **View scale** list, choose **Manual**.
4. In the **y scale** text field, type 40.
5. Click **Update**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.

MATERIALS

The cavity is filled with air.

ADD MATERIAL

1 In the **Materials** toolbar, click **Add Material** to open the **Add Material** window.

2 Go to the **Add Material** window.

3 In the tree, select **Built-in** > **Air**.

4 Click the **Add to Component** button in the window toolbar.

5 In the **Materials** toolbar, click **Add Material** to close the **Add Material** window.

ELECTROMAGNETIC WAVES, BEAM ENVELOPES (EWBE)

1 In the **Settings** window for **Electromagnetic Waves, Beam Envelopes**, locate the **Components** section.

2 From the **Electric field components solved for** list, choose **Out-of-plane vector**.

ADD PHYSICS

Add an ODE interface to solve for the frequency.

1 In the **Physics** toolbar, click **Add Physics** to open the **Add Physics** window.

2 Go to the **Add Physics** window.

3 In the tree, select **Mathematics** > **ODE and DAE Interfaces** > **Global ODEs and DAEs (ge)**.

4 Click the **Add to Component 1** button in the window toolbar.

5 In the **Home** toolbar, click **Add Physics** to close the **Add Physics** window.

GLOBAL ODES AND DAES (GE)

Global Equations 1 (ODE1)

1 In the **Settings** window for **Global Equations**, locate the **Global Equations** section.

2 In the table, enter the following settings:

Name	f(u,ut,utt,t) (1)	Initial value (u_0) (1)	Initial value (ut_0) (1/s)	Description
freq1	1-nEz	fqn	0	Frequency

The equation above represents a normalization of the electric field. The normalization variable nEz will be defined below.

3 Locate the **Units** section. Click **Select Dependent Variable Quantity**.

4 In the **Physical Quantity** dialog, type frequency in the text field.

5 In the tree, select **General** > **Frequency (Hz)**.

6 Click **OK**.

7 In the **Settings** window for **Global Equations**, locate the **Units** section.

8 Click **Define Source Term Unit**.

9 In the **Source term quantity** table, enter the following settings:

Source term quantity	Unit
Custom unit	V^2/m^2

ELECTROMAGNETIC WAVES, BEAM ENVELOPES (EWBE)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Electromagnetic Waves, Beam Envelopes (ewbe)**.

2 In the **Settings** window for **Electromagnetic Waves, Beam Envelopes**, click to expand the **Equation** section.

3 From the **Equation form** list, choose **Frequency domain**.

4 From the **Frequency** list, choose **User defined**. In the *f* text field, type freq1.

Now the electric field will be solved self-consistently for the frequency solved for by the **Global Equations 1** node.

Initial Values 1

Insert an approximation of the electric field that makes the nonlinear solver converge to the correct mode fields.

1 In the **Model Builder** window, under **Component 1 (comp1)** > **Electromagnetic Waves, Beam Envelopes (ewbe)** click **Initial Values 1**.

2 In the **Settings** window for **Initial Values**, locate the **Initial Values** section.

3 Specify the $\mathbf{E1}$ vector as

`0`	x
`0`	y
`hermite(n,sqrt(2)*y/w0)*exp(-(y/w0)^2)`	z

4 Specify the $\mathbf{E2}$ vector as

`0`	x
`0`	y
`-hermite(n,sqrt(2)*y/w0)*exp(-(y/w0)^2)*exp(-j*k0*d)`	z

DEFINITIONS

Now define the integration operator and the variables for normalization of the electric field.

Integration 1 (intop1)

1 In the **Definitions** toolbar, click **Nonlocal Couplings** and choose **Integration**.

2 Select Domain 1 only.

Variables 1

1 In the **Model Builder** window, right-click **Definitions** and choose **Variables**.

2 In the **Settings** window for **Variables**, locate the **Variables** section.

3 In the table, enter the following settings:

Name	**Expression**	**Unit**	**Description**
A	`intop1(1)`	m^2	Area
nEz	`intop1(realdot(ewbe.Ez,ewbe.Ez))/A`	$kg^2 \cdot m^2/(s^6 \cdot A^2)$	Normalization

MESH 1

Use physics-controlled meshing to create a mapped mesh that resolves the spot radius in the transverse dimension and the Rayleigh range in the longitudinal dimension.

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mesh 1**.

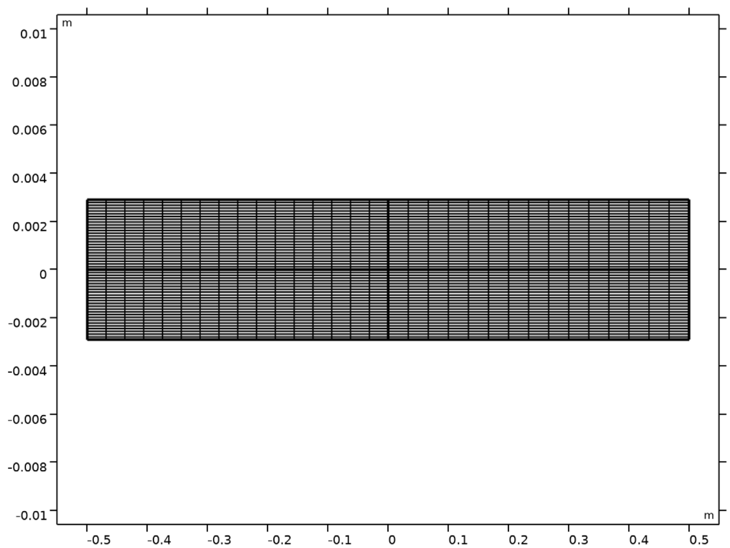
2 In the **Settings** window for **Mesh**, locate the **Electromagnetic Waves, Beam Envelopes (ewbe)** section.

3 In the N_T text field, type 50.

4 In the N_L text field, type 30.

5 In the **Mesh** toolbar, click **Build Mesh**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.



STUDY 1

Create a parametric sweep, to look for the fundamental mode and higher-order transverse modes.

Parametric Sweep

1 In the **Study** toolbar, click **Parametric Sweep**.

2 In the **Settings** window for **Parametric Sweep**, locate the **Study Settings** section.

3 Click **Add**.

4 In the table, enter the following settings:

Parameter name	Parameter value list	Parameter unit
n (Transverse mode number)	0 1 2	

Solution 1 (sol1)

Solve the stationary problem using complex splitting, to split the complex expressions into their real and imaginary parts. Thereby the expressions become analytical and a correct Jacobian can be calculated. This makes the problem solve faster and more robustly.

1 In the **Study** toolbar, click **Show Default Solver**.
2 In the **Model Builder** window, expand the **Solution 1 (sol1)** node, then click **Compile Equations: Stationary**.
3 In the **Settings** window for **Compile Equations**, locate the **Study and Step** section.
4 Select the **Split complex variables in real and imaginary parts** checkbox.
5 Click the **Show More Options** button in the **Model Builder** toolbar.
6 In the **Show More Options** dialog, in the tree, select the checkbox for the node **Physics** > **Advanced Physics Options**.
7 Click **OK**.

GLOBAL ODES AND DAES (GE)

Global Equations 1 (ODE1)

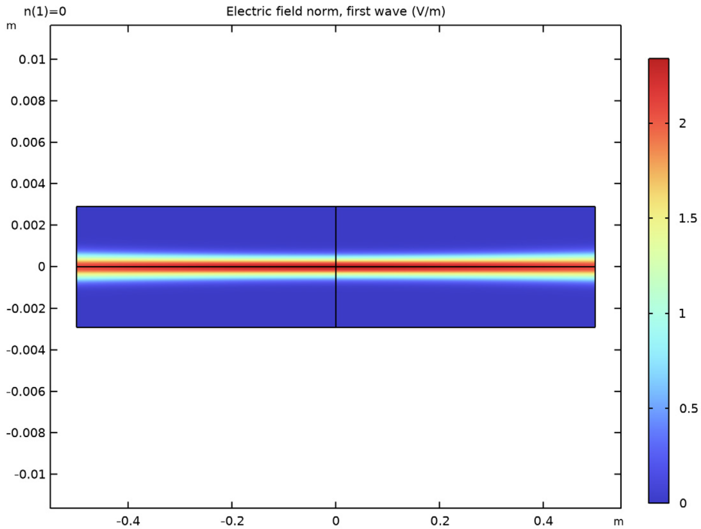
1 In the **Model Builder** window, under **Component 1 (comp1)** > **Global ODEs and DAEs (ge)** click **Global Equations 1 (ODE1)**.
2 In the **Settings** window for **Global Equations**, click to expand the **Discretization** section.
3 From the **Value type when using splitting of complex variables** list, choose **Real**.
4 In the **Study** toolbar, click **Compute**.

RESULTS

Electric Field, First Wave (ewbe)

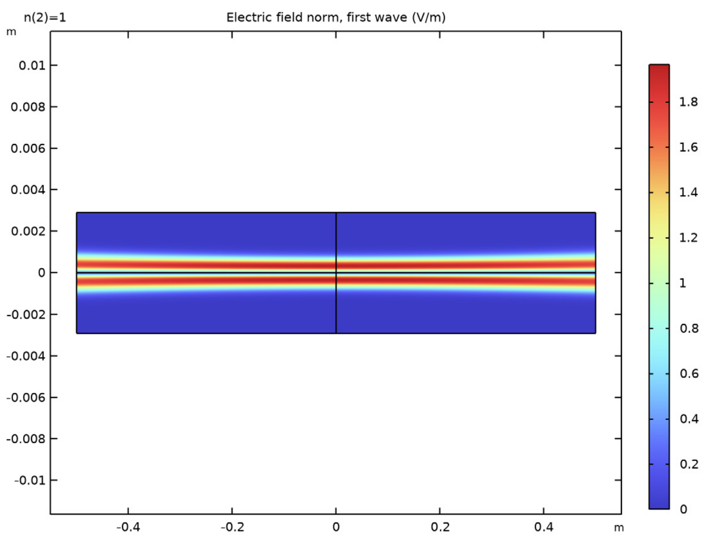
1 In the **Settings** window for **2D Plot Group**, locate the **Data** section.
2 From the **Parameter value (n)** list, choose **0**.
3 In the **Electric Field, First Wave (ewbe)** toolbar, click **Plot**.

4 Click the **Zoom Extents** button in the **Graphics** toolbar.



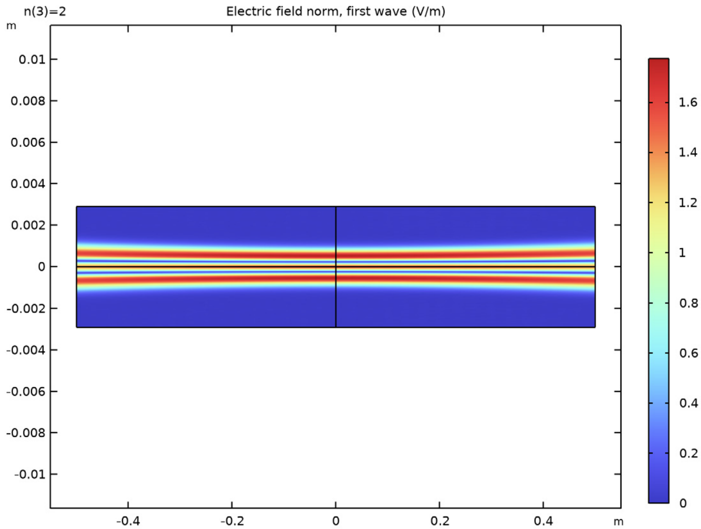
5 From the **Parameter value (n)** list, choose **1**.

6 In the **Electric Field, First Wave (ewbe)** toolbar, click **Plot**.



7 From the **Parameter value (n)** list, choose **2**.

8 In the **Electric Field, First Wave (ewbe)** toolbar, click **Plot**.



Global Evaluation 1

Now compare the analytical mode frequencies to the computed ones.

1 In the **Model Builder** window, expand the **Results** > **Derived Values** node, then click **Global Evaluation 1**.

2 In the **Settings** window for **Global Evaluation**, click **Evaluate**.

3 Locate the **Expressions** section. In the table, enter the following settings:

Expression	**Unit**	**Description**
`fqn`	`Hz`	`Mode frequency`

4 Click **Evaluate**.

TABLE 1

1 Go to the **Table 1** window.

2 Click the **Full Precision** button in the window toolbar. At least the first six to seven digits should be the same for the computed and the analytical mode frequencies.