

Model created in COMSOL Multiphysics 6.4

Uncertainty Quantification of the Ishigami Function

Introduction

This example demonstrates how to perform uncertainty quantification analysis of the Ishigami function. This random function of three variables is a well-known benchmark used to test global sensitivity analysis and uncertainty quantification algorithms. The mean, standard deviation, maximum, and minimum values as well as Sobol indices of the Ishigami function can be calculated analytically for the input distributions used here.

For this test problem, the Ishigami function is

$$f(X_1, X_2, X_3) = \sin(X_1) + a(\sin(X_2))^2 + bX_3^4 \sin(X_1)$$

where X_1, X_2 , and X_3 are independent uniformly distributed random variables in $[-\pi, +\pi]$ with $a = 7$ and $b = 0.1$.

The function can be visualized in 3D by using, for example, a slice plot as in [Figure 1](#).

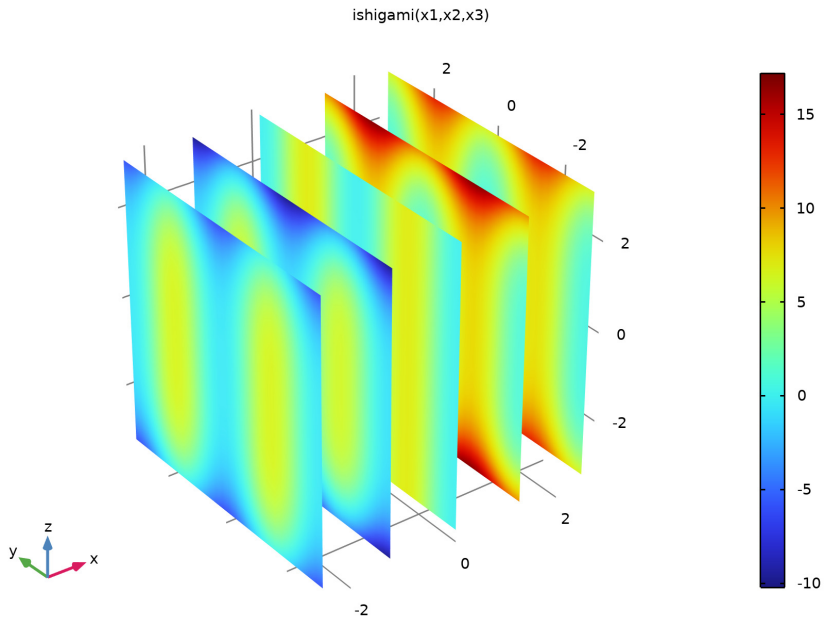


Figure 1: Slice plot of the Ishigami function.

The analytically computed values are according to [Table 1](#).

TABLE 1: ANALYTICAL BENCHMARK VALUES.

QUANTITY	EXPRESSION	NUMERICAL VALUE (ROUNDED)
Mean value	$a/2$	3.5
Variance (V)	$(a^2)/8 + b*(\pi^4)/5 + b^2*(\pi^8)/18 + 1/2$	13.845
Maximum	$8 + (\pi^4)/10$	17.741
Minimum	$-1 - (\pi^4)/10$	-10.741
Standard deviation	\sqrt{V}	3.7208
First-order Sobol index X_1	$(0.5*(1+b*(\pi^4)/5)^2)/V$	0.31391
First-order Sobol index X_2	$((a^2)/8)/V$	0.44241
First-order Sobol index X_3	0	0
Total Sobol index X_1	$((1/2)*(1+b*(\pi^4)/5)^2 + (8*b^2*\pi^8)/225)/V$	0.55759
Total Sobol index X_2	$((a^2)/8)/V$	0.44241
Total Sobol index X_3	$((8*b^2*\pi^8)/225)/V$	0.24368

For reference, these values are entered as global parameters in the model.

Model Definition

The model runs through 3 uncertainty quantification studies: **Screening**, **Sensitivity analysis**, and **Uncertainty Propagation** using the Ishigami function as the quantity of interest. In order to perform the uncertainty quantification analysis, the three random variables need to be defined as global parameters using arbitrary values. The actual values

for these variables will, during the simulation, be randomized by the uncertainty quantification algorithms. All the global parameters in the model are shown in [Figure 2](#).

Name	Expression	Value	Description
X1	1	1	Random variable 1
X2	1	1	Random variable 2
X3	1	1	Random variable 3
a	7	7	Ishigami parameter 1
b	0.1	0.1	Ishigami parameter 2
M	$a/2$	3.5	Mean
V	$(a^2)/8 + b*(\pi^4)/5 + b^2*(\pi^8)/18 + 1/2$	13.845	Variance
STD	\sqrt{V}	3.7208	Standard deviation
V1	$0.5*(1 + b*(\pi^4)/5)^2$	4.3459	First-order variance 1
V2	$(a^2)/8$	6.125	First-order variance 2
V3	0	0	First-order variance 3
S1	$V1/V$	0.31391	First-order Sobol index 1
S2	$V2/V$	0.44241	First-order Sobol index 2
S3	$V3/V$	0	First-order Sobol index 3
VT1	$(1/2)*(1 + b*(\pi^4)/5)^2 + (8*b^2*\pi^8)/225$	7.7196	Total variance 1
VT2	$(a^2)/8$	6.125	Total variance 2
VT3	$(8*b^2*\pi^8)/225$	3.3737	Total variance 3
ST1	$VT1/V$	0.55759	Total Sobol index 1
ST2	$VT2/V$	0.44241	Total Sobol index 2
ST3	$VT3/V$	0.24368	Total Sobol index 3
imax	$8 + (\pi^4)/10$	17.741	Function max
imin	$-1 - (\pi^4)/10$	-10.741	Function min

Figure 2: The model parameters.

The Ishigami function is defined as an analytic function with three input arguments as shown in Figure 3.

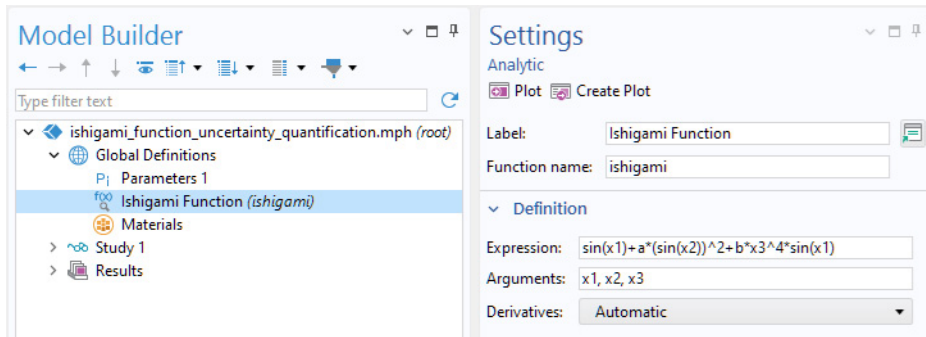


Figure 3: The Ishigami function entered as an Analytic function, ishigami.

Results and Discussion

The sensitivity analysis shows that the computed Sobol indices are consistent with the true analytical values, as shown in Figure 4 below.

	ishigami(X1,X2,X3), First order	ishigami(X1,X2,X3), Total
X1	0.31363	0.55131
X2	0.44844	0.44870
X3	8.2735E-5	0.23785

Figure 4: The computed Sobol indices.

Similarly, the values for mean, standard deviation (STD), minimum, and maximum are consistent with the analytical values, as shown in Figure 5.

	Mean	STD	Minimum	Maximum	Lower 90%	Upper 90%	Lower 95%	Upper 95%	Lower 99%	Upper 99%
ishigami(X1,X2,X3)	3.5186	3.6494	-10.094	17.337	-2.0346	9.2073	-3.6171	10.986	-6.8413	13.770

Figure 5: The computed values for mean, standard deviation, minimum, maximum, and confidence intervals.

The accuracy of the results can be increased by lowering tolerances or increasing the number of sample input points.

The computed kernel density estimation is displayed in [Figure 6](#).

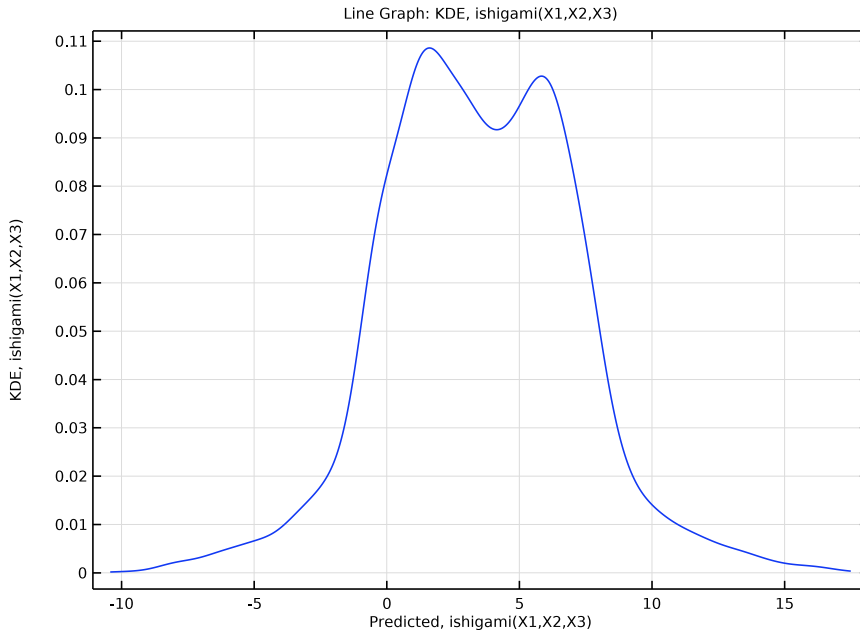


Figure 6: The KDE plot for the Ishigami function.

These uncertainty quantification results can be compared not only with the analytical values but also with that of the direct Monte Carlo simulation performed in the model [Direct Monte Carlo Simulation of the Ishigami Function](#).

Reference


1. T. Ishigami and T. Homma, "An importance quantification technique in uncertainty analysis for computer models," *Proc. First Int'l Symp. Uncertainty Modeling and Analysis*, IEEE, pp. 398-403, 1990.

Application Library path: `Uncertainty_Quantification_Module/Tutorials/ishigami_function_uncertainty_quantification`



Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click  **Blank Model**.

ADD STUDY

- 1 In the **Home** toolbar, click  **Add Study** to open the **Add Study** window.
- 2 Go to the **Add Study** window.
- 3 Find the **Studies** subsection. In the **Select Study** tree, select **Preset Studies for Selected Physics Interfaces > Stationary**.
- 4 Click the **Add Study** button in the window toolbar.
- 5 In the **Home** toolbar, click  **Add Study** to close the **Add Study** window.

GLOBAL DEFINITIONS


Parameters |

- 1 In the **Settings** window for **Parameters**, locate the **Parameters** section.
- 2 In the table, enter the following settings:

Name	Expression	Value	Description
X1	1	1	Random variable 1
X2	1	1	Random variable 2
X3	1	1	Random variable 3
a	7	7	Ishigami parameter 1
b	0.1	0.1	Ishigami parameter 2
M	$a/2$	3.5	Mean
V	$(a^2)/8 + b*(\pi^4)/5 + b^2*(\pi^8)/18 + 1/2$	13.845	Variance
STD	\sqrt{V}	3.7208	Standard deviation
V1	$0.5*(1 + b*(\pi^4)/5)^2$	4.3459	First-order variance 1
V2	$(a^2)/8$	6.125	First-order variance 2
V3	0	0	First-order variance 3
S1	V1/V	0.31391	First-order Sobol index 1

Name	Expression	Value	Description
S2	$V2/V$	0.44241	First-order Sobol index 2
S3	$V3/V$	0	First-order Sobol index 3
VT1	$(1/2)*(1+b*(\pi^4)/5)^2+(8*b^2*\pi^8)/225$	7.7196	Total variance 1
VT2	$(a^2)/8$	6.125	Total variance 2
VT3	$(8*b^2*\pi^8)/225$	3.3737	Total variance 3
ST1	$VT1/V$	0.55759	Total Sobol index 1
ST2	$VT2/V$	0.44241	Total Sobol index 2
ST3	$VT3/V$	0.24368	Total Sobol index 3
imax	$8+(\pi^4)/10$	17.741	Function max
imin	$-1-(\pi^4)/10$	-10.741	Function min

Ishigami Function

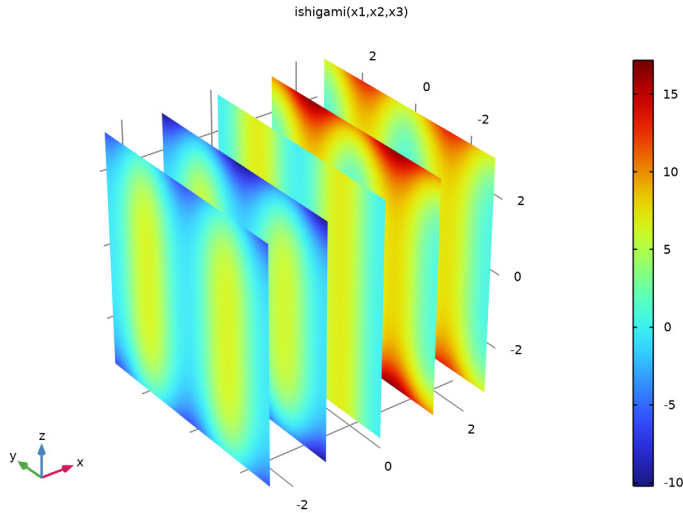
- 1 In the **Home** toolbar, click  **Functions** and choose **Global > Analytic**.
- 2 In the **Settings** window for **Analytic**, type **ishigami** in the **Function name** text field.
- 3 Locate the **Definition** section. In the **Expression** text field, type $\sin(x1)+a*(\sin(x2))^2+b*x3^4*\sin(x1)$.
- 4 In the **Arguments** text field, type $x1, x2, x3$.
- 5 In the **Label** text field, type **Ishigami Function**.
- 6 Locate the **Plot Parameters** section. In the table, enter the following settings:

Plot	Argument	Lower limit	Upper limit	Fixed value	Unit
√	x1	-pi	pi	0	
√	x2	-pi	pi	0	
√	x3	-pi	pi	0	

- 7 Click  **Create Plot**.


RESULTS

3D Plot Group 1



STUDY 1

Uncertainty Quantification

- 1 In the **Study** toolbar, click  **More Study Extensions** and choose **Uncertainty Quantification**.
- 2 In the **Settings** window for **Uncertainty Quantification**, locate the **Quantities of Interest** section.
- 3 Click **+ Add**.
- 4 In the table, enter the following settings:

Expression	Include study-dependent input
ishigami(X1,X2,X3)	Reduce to single global output

- 5 Locate the **Input Parameters** section. Click **+ Add** three times.
- 6 In the table, click to select the cell at row number 1 and column number 1.
- 7 In the **Lower bound** text field, type $-\pi$.
- 8 In the **Upper bound** text field, type π .
- 9 In the table, click to select the cell at row number 2 and column number 1.


I0 In the **Lower bound** text field, type -pi.

I1 In the **Upper bound** text field, type pi.

I2 In the table, click to select the cell at row number 3 and column number 1.

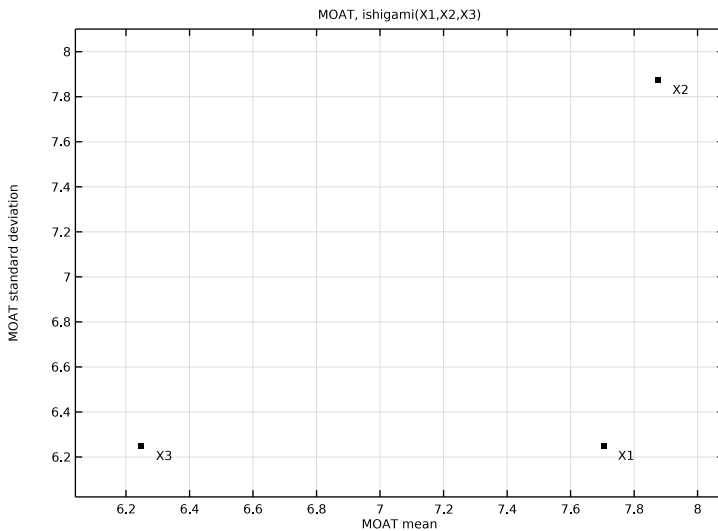
I3 In the **Lower bound** text field, type -pi.

I4 In the **Upper bound** text field, type pi.

I5 In the **Study** toolbar, click  **Compute**.

RESULTS

MOAT, ishigami(X1,X2,X3)



The screening study indicates that, while all parameters are influential, X3 and X2 stand out as the most critical ones.

STUDY 1

Uncertainty Quantification


In the **Model Builder** window, under **Study 1** right-click **Uncertainty Quantification** and choose **Add New Uncertainty Quantification Study For > Sensitivity Analysis**.

STUDY 2

To achieve a high level of accuracy, configure the surrogate model to use a lower relative tolerance.

- 1 In the **Model Builder** window, under **Study 2** click **Uncertainty Quantification**.
- 2 In the **Settings** window for **Uncertainty Quantification**, locate the **Uncertainty Quantification Settings** section.
- 3 Find the **Surrogate model settings** subsection. In the **Relative tolerance** text field, type 0.0005.

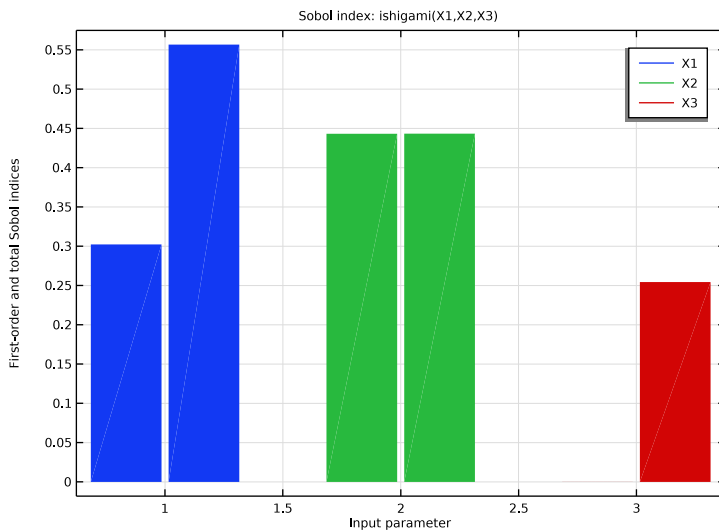
Uncertainty Quantification 2

In the **Study** toolbar, click  **Compute**.

RESULTS

Sobol Index, QoI

- 1 In the **Model Builder** window, under **Results** > **Uncertainty Quantification Graph 1** click **Sobol Index, QoI**.



The Sensitivity analysis study computes Sobol indices that are consistent with the analytical values.

STUDY 2

Uncertainty Quantification

In the **Model Builder** window, under **Study 2** right-click **Uncertainty Quantification** and choose **Add New Uncertainty Quantification Study For > Uncertainty Propagation**.

STUDY 3


Now, change the **Surrogate model** to **Adaptive sparse polynomial chaos expansion**. For the Ishigami function, the polynomial chaos expansion surrogate model turns out to be much more efficient than the default **Adaptive Gaussian process** option.

- 1 In the **Model Builder** window, under **Study 3** click **Uncertainty Quantification**.
- 2 In the **Settings** window for **Uncertainty Quantification**, locate the **Uncertainty Quantification Settings** section.
- 3 Find the **Surrogate model settings** subsection. From the **Surrogate model** list, choose **Adaptive sparse polynomial chaos expansion**.

Since the surrogate model has already been built from the Sensitivity analysis study, we can reuse the surrogate function, change to **Analyze only**.

- 4 From the **Compute action** list, choose **Analyze only**.

Uncertainty Quantification 3

In the **Study** toolbar, click  **Compute**.

RESULTS

Kernel Density Estimation, pce_l_1

In the **Model Builder** window, under **Results > Uncertainty Quantification Graph 2** click **Kernel Density Estimation, pce_l_1**.

