Model created in COMSOL Multiphysics 6.4

Ideal Cloak[1]

1. This model is courtesy of Yaroslav Urzhumov, Center for Metamaterials and Integrated Plasmonics, Duke University Durham, NC.

Introduction

Electromagnetic or optical invisibility can be achieved by coating an object with a transparent gradient-index structure that bends the rays of light around the concealed object (Ref. 1). The structure has to be able to do so with radiation incident from any direction, which can be achieved by making it rotationally invariant. Inside this transparent shell, situated is a nontransparent object whose scattering properties in free space are inessential as long as the cloak operates perfectly by preventing any light rays from hitting the object. The concept of invisibility based on omnidirectional cloaking was introduced by Sir John Pendry (Imperial College, UK) and his collaborators in 2006 (Ref. 1).

The cloak of invisibility modeled here is a concentric spherical shell, whose interior surface represents the concealed cavity. Omnidirectional cloaks require anisotropic material properties, which can be calculated using the transformation optics theory (Ref. 1). Although ray and beam bending can be achieved with a gradient of isotropic refractive index, index gradient alone is not sufficient for omnidirectional invisibility. This can be shown by means of the uniqueness theorem that applies to scattering problems involving bodies composed of isotropic materials (Ref. 2).

The refractive index in the azimuthal directions (normal to the radial direction) experiences a gradual change from unity on the exterior surface of the cloak, where it matches free space, down to zero on the interior surface. With a proper choice of index distribution, you can ensure that any ray hitting the cloak never reaches the interior surface, and thus never probes the object. The refractive index in the radial direction is not continuous in this particular cloak design. The resulting index discontinuity at the exterior surface does not lead to reflections because only the index in the tangential direction affects reflectivity.

This model demonstrates the use of optical tracing for studying optically large gradient-index structures with anisotropic optical properties. Additionally, the model introduces a smoothing technique for handling discontinuities of refractive index on curved surfaces, which are typical in conventional optical devices such as lenses.

Model Definition

There is no explicit support for modeling geometrical optics in the Particle Tracing Module, but an analogy between the Hamilton equations and the equations for rays in the zero wavelength limit allows us to solve the problem. The analogy is as follows:

- The wave vector, $\mathbf{k}$ (SI unit: 1/m) plays the same role in geometrical optics as the momentum, $\mathbf{p}$, of particles in classical mechanics.

- The angular frequency, ω (SI unit: 1/s) plays the role of the Hamiltonian, H.

  For a classical particle, Hamilton's equations are:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\frac{\partial H}{\partial t}, \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial H}{\partial \mathbf{p}}$$

  and using the analogy above:

$$\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}t} = -\frac{\partial \omega}{\partial t}, \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial \omega}{\partial \mathbf{k}}$$

- The particle mass should be set to 1.

For geometrical optics, the angular frequency is given by

$$\omega = \frac{c|\mathbf{k}|}{n}$$

where n (dimensionless) is the refractive index of the material and c (SI unit: m/s) is the speed of light in a vacuum. For vacuum, the refractive index is simply 1. Inside the cloak, the refractive index is anisotropic, so it is more convenient to express the wave vector using spherical coordinates:

$$\begin{bmatrix} k_r \\ k_\vartheta \\ k_\phi \end{bmatrix} = \begin{bmatrix} \sin\vartheta\cos\phi & \sin\vartheta\sin\phi & \cos\vartheta \\ \cos\vartheta\cos\phi & \cos\vartheta\sin\phi & -\sin\vartheta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$$

The angular frequency is hence given by:

$$\omega = c\left(\frac{k_r^2}{n_r^2} + \frac{k_\vartheta^2}{n_\vartheta^2} + \frac{k_\phi^2}{n_\phi^2}\right)^{\frac{1}{2}}$$

Results and Discussion

The ray trajectories are plotted in Figure 1. The rays reach the cloak and bend around the inner sphere, which would appear invisible to an observer.

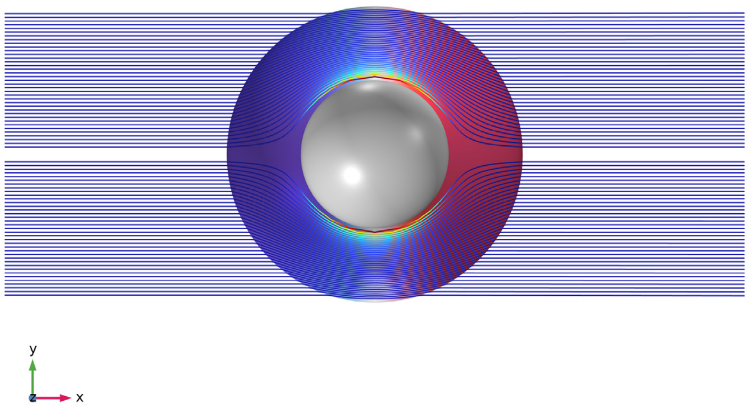

Figure 1: Plot of the light rays traveling through the cloak.

A better way of determining whether the incoming rays are returned to their original trajectory is by using a **Poincaré Map** or **Phase Portrait** plot. Figure 2 shows a **Poincaré Map** in the *yz*-plane at the initial time step (red dots) and at the final time step (blue dots). In this Poincaré map, the horizontal position indicates the particle's *y*-coordinate and the vertical position indicates the particle's *z*-coordinate. The particle position after traveling through the cloak is almost exactly the same as it was initially.

Figure 3 shows the change in the particles' position in the *yz*-plane after traveling through the cloaking device. The particles at the maximum and minimum *y*-coordinates have greater absolute error in their final positions despite being deflected at lower angles compared to the particles passing through the middle of the cloak. The higher absolute error may be due to these particles entering the anisotropic domain at a very oblique angle of incidence.

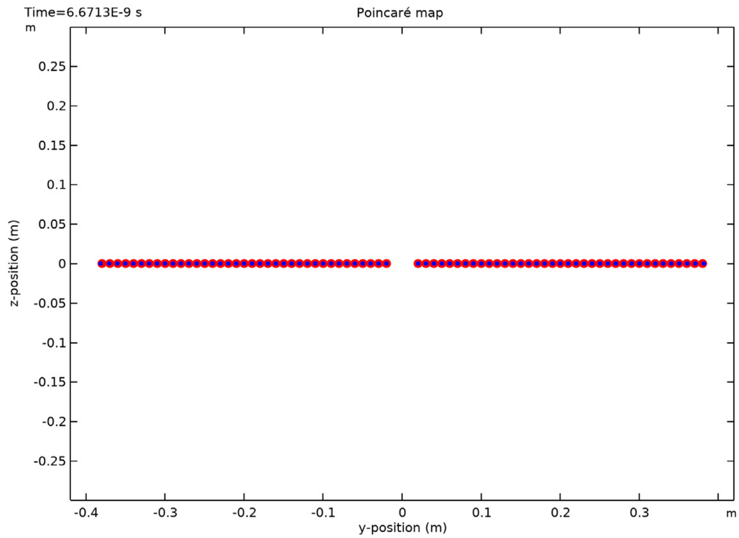

Figure 2: Poincaré map in the yz-plane at for Poincaré sections at $x = -1$ (red) and $x = 1$ (blue).

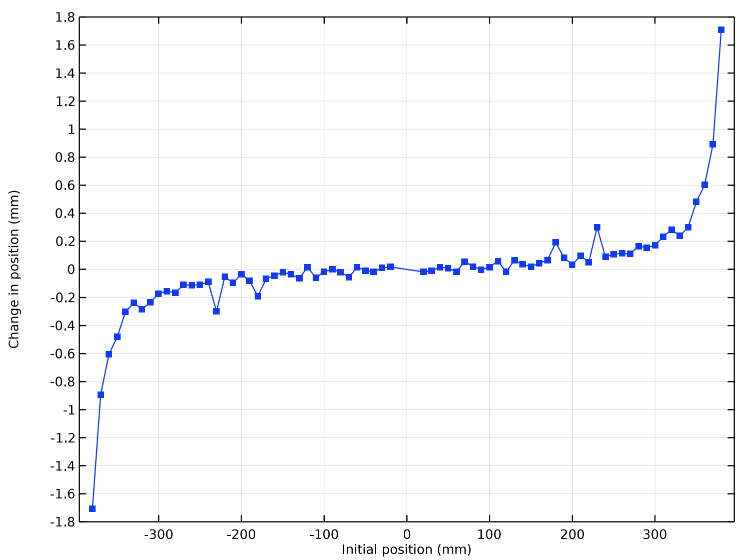

Figure 3: Change in the y-component of the particle position after traveling through the cloaking device.

References

1. J. Pendry, D. Schurig, and D.R. Smith, “Controlling Electromagnetic Fields,” *Science*, vol. 312, no. 5781, pp. 1780–1782, 2006.

2. A.I. Nachman, “Reconstruction from Boundary Measurements,” *Ann. Math.*, vol. 128, pp. 531–576, 1988.

Application Library path: `Particle_Tracing_Module/Tutorials/ideal_cloak`

Modeling Instructions

This model comes courtesy of Yaroslav Urzhumov, Center for Metamaterials and Integrated Plasmonics, Duke University.

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **3D**.

2 In the **Select Physics** tree, select **Mathematics** > **Mathematical Particle Tracing (pt)**.

3 Click **Add**.

4 Click **Study**.

5 In the **Select Study** tree, select **General Studies** > **Time Dependent**.

6 Click **Done**.

GLOBAL DEFINITIONS

Specify the dimensions of the air domain and the cloak.

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

Name	Expression	Value	Description
L	1[m]	1 m	Box length
a	0.2[m]	0.2 m	Inner radius
b	0.4[m]	0.4 m	Outer radius
n_air	1	1	Refractive index of air
hmax	L*0.2	0.2 m	Maximum element size in volume
hmax_cloak	b*0.05	0.02 m	Maximum element size in cloak

GEOMETRY 1

Block 1 (blk1)

1. In the **Geometry** toolbar, click **Block**.
2. In the **Settings** window for **Block**, locate the **Size and Shape** section.
3. In the **Width** text field, type `2*L`.
4. In the **Depth** text field, type `2*L`.
5. In the **Height** text field, type `2*L`.
6. Locate the **Position** section. From the **Base** list, choose **Center**.

Sphere 1 (sph1)

1. In the **Geometry** toolbar, click **Sphere**.
2. In the **Settings** window for **Sphere**, locate the **Size** section.
3. In the **Radius** text field, type `a`.

Sphere 2 (sph2)

1. In the **Geometry** toolbar, click **Sphere**.
2. In the **Settings** window for **Sphere**, locate the **Size** section.
3. In the **Radius** text field, type `b`.
4. Click **Build All Objects**.
5. Click the **Go to Default View** button in the **Graphics** toolbar.

DEFINITIONS

Now add the expressions which transform the refractive index of the cloak from Cartesian to spherical coordinates. The wave vector must also be transformed.

Variables 1

1 In the **Definitions** toolbar, click a= **Local Variables**.

Load the variable definitions from a file.

2 In the **Settings** window for **Variables**, locate the **Variables** section.

3 Click **Load from File**.

4 Browse to the model's Application Libraries folder and double-click the file `ideal_cloak_variables.txt`.

MATHEMATICAL PARTICLE TRACING (PT)

Using the analogy presented in the introduction section above, enter an expression for the angular frequency, which is called `H_photon` in this case.

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Mathematical Particle Tracing (pt)**.

2 In the **Settings** window for **Mathematical Particle Tracing**, locate the **Particle Release and Propagation** section.

3 From the **Formulation** list, choose **Hamiltonian**.

Particle Properties 1

1 In the **Model Builder** window, under **Component 1 (comp1)** > **Mathematical Particle Tracing (pt)** click **Particle Properties 1**.

2 In the **Settings** window for **Particle Properties**, locate the **Hamiltonian** section.

3 In the H text field, type `H_photon`.

4 Locate the **Particle Mass** section. In the m_p text field, type `1`.

Release from Grid 1

Next, release the particles in the y direction for fixed x- and z-coordinates.

1 In the **Physics** toolbar, click **Global** and choose **Release from Grid**.

2 In the **Settings** window for **Release from Grid**, locate the **Initial Coordinates** section.

3 In the $q_{x,0}$ text field, type `-1`.

4 In the $q_{y,0}$ text field, type `range(-0.38,0.01,-0.02) range(0.02,0.01,0.38)`.

5 Locate the **Initial Velocity** section. Specify the $\mathbf{v}_0$ vector as

1	x
0	y
0	z

Because the Hamiltonian formulation is being used to model rays, the settings entered for the **Initial velocity** determine the initial wave vector direction, not the velocity of the model particles.

MESH 1

The mesh needs to be fine in the cloak region so that the particle trajectories can be computed to a high degree of accuracy.

Free Tetrahedral 1

In the **Mesh** toolbar, click **Free Tetrahedral**.

Size 1

1 Right-click **Free Tetrahedral 1** and choose **Size**.

2 In the **Settings** window for **Size**, locate the **Geometric Entity Selection** section.

3 From the **Geometric entity level** list, choose **Domain**.

4 Select Domain 2 only.

5 Click to expand the **Element Size Parameters** section. Locate the **Element Size** section. Click the **Custom** button.

6 Locate the **Element Size Parameters** section.

7 Select the **Maximum element size** checkbox. In the associated text field, type `hmax_cloak`.

8 Select the **Minimum element size** checkbox. In the associated text field, type `hmax_cloak/2`.

Size

1 In the **Model Builder** window, under **Component 1 (comp1)** > **Mesh 1** click **Size**.

2 In the **Settings** window for **Size**, locate the **Element Size** section.

3 Click the **Custom** button.

4 Locate the **Element Size Parameters** section. In the **Maximum element size** text field, type `hmax`.

5 In the **Minimum element size** text field, type `hmax/2`.

6 In the **Model Builder** window, right-click **Mesh 1** and choose **Build All**.

STUDY 1

To accurately compute the particle trajectories in an anisotropic medium, the default solver tolerances need to be made more strict. Do this by first showing the default solver, and then reducing the relative and absolute tolerances.

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 Click **Range**.

4 In the **Range** dialog, type `2[m]/c_const` in the **Stop** text field.

5 From the **Entry method** list, choose **Number of values**.

6 In the **Number of values** text field, type `301`.

7 Click **Replace**.

8 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

9 From the **Tolerance** list, choose **User controlled**.

10 In the **Relative tolerance** text field, type `1e-6`.

Solution 1 (sol1)

1 In the **Study** toolbar, click **Show Default Solver**.

2 In the **Model Builder** window, expand the **Solution 1 (sol1)** node.

3 In the **Model Builder** window, under **Study 1** > **Solver Configurations** > **Solution 1 (sol1)** click **Time-Dependent Solver 1**.

4 In the **Settings** window for **Time-Dependent Solver**, click to expand the **Time Stepping** section.

5 From the **Solver type** list, choose **Explicit**.

6 From the **Runge–Kutta method** list, choose **Dormand–Prince 5**.

7 In the **Study** toolbar, click **Compute**.

RESULTS

Particle Trajectories (pt)

The path of the rays is best visualized by adding selections for the cloak inner and outer surfaces.

1 In the **Settings** window for **3D Plot Group**, locate the **Plot Settings** section.

2 Clear the **Plot dataset edges** checkbox.

3 Click to expand the **Title** section. From the **Title type** list, choose **None**.

4 In the **Model Builder** window, expand the **Particle Trajectories (pt)** node.

Particle Trajectories 1

1 In the **Model Builder** window, expand the **Results** > **Particle Trajectories (pt)** > **Particle Trajectories 1** node, then click **Particle Trajectories 1**.

2 In the **Settings** window for **Particle Trajectories**, locate the **Coloring and Style** section.

3 Find the **Line style** subsection. From the **Type** list, choose **Line**.

4 Find the **Point style** subsection. From the **Type** list, choose **None**.

Color Expression 1

1 In the **Model Builder** window, click **Color Expression 1**.

2 In the **Settings** window for **Color Expression**, locate the **Coloring and Style** section.

3 Clear the **Color legend** checkbox.

Surface 1

1 In the **Model Builder** window, right-click **Particle Trajectories (pt)** and choose **Surface**.

2 In the **Settings** window for **Surface**, locate the **Data** section.

3 From the **Dataset** list, choose **Study 1/Solution 1 (sol1)**.

4 Locate the **Expression** section. In the **Expression** text field, type `cos_phi`.

5 Locate the **Coloring and Style** section. From the **Color table** list, choose **WaveLight**.

6 Clear the **Color legend** checkbox.

Selection 1

1 Right-click **Surface 1** and choose **Selection**.

2 Click the **Wireframe Rendering** button in the **Graphics** toolbar.

3 Select Boundaries 6, 8, 14, and 18 only.

Surface 2

1 In the **Model Builder** window, right-click **Particle Trajectories (pt)** and choose **Surface**.

2 In the **Settings** window for **Surface**, locate the **Data** section.

3 From the **Dataset** list, choose **Study 1/Solution 1 (sol1)**.

4 Locate the **Coloring and Style** section. From the **Coloring** list, choose **Uniform**.

5 From the **Color** list, choose **Gray**.

Selection 1

1 Right-click **Surface 2** and choose **Selection**.

2 In the **Settings** window for **Selection**, locate the **Selection** section.

3 From the **Geometric entity level** list, choose **Domain**.

4 Select Domain 3 only.

5 In the **Particle Trajectories (pt)** toolbar, click **Plot**.

DEFINITIONS

View 1

1 In the **Model Builder** window, under **Component 1 (comp1)** > **Definitions** click **View 1**.

2 In the **Settings** window for **View**, locate the **View** section.

3 Clear the **Show grid** checkbox.

RESULTS

Particle Trajectories 1

1 In the **Model Builder** window, under **Results** > **Particle Trajectories (pt)** click **Particle Trajectories 1**.

2 In the **Particle Trajectories (pt)** toolbar, click **Plot**.

3 Click the **Go to XY View** button in the **Graphics** toolbar. The plot should look like Figure 1.

To see how well the cloak performs, look at the rays in phase space before and after they pass through the cloak. You can do this in two different ways.

The first method is to define a pair of **Cut Plane** datasets on the incoming and outgoing sides of the cloak, then plot a **Poincaré Map** of the ray positions as they intersect each plane.

Cut Plane 1

1 In the **Model Builder** window, expand the **Results** > **Datasets** node.

2 Right-click **Results** > **Datasets** and choose **Cut Plane**.

3 In the **Settings** window for **Cut Plane**, locate the **Data** section.

4 From the **Dataset** list, choose **Particle 1**.

5 Locate the **Plane Data** section. In the **x-coordinate** text field, type `-0.99`.

Cut Plane 2

1 Right-click **Cut Plane 1** and choose **Duplicate**.

2 In the **Settings** window for **Cut Plane**, locate the **Plane Data** section.

3 In the **x-coordinate** text field, type 0.99.

Ray Position Relative to Initial Position

1 In the **Results** toolbar, click **2D Plot Group**.

2 In the **Settings** window for **2D Plot Group**, type Ray Position Relative to Initial Position in the **Label** text field.

3 Locate the **Plot Settings** section.

4 Select the **x-axis label** checkbox. In the associated text field, type y-position (m).

5 Select the **y-axis label** checkbox. In the associated text field, type z-position (m).

Poincaré Map 1

1 In the **Ray Position Relative to Initial Position** toolbar, click **More Plots** and choose **Poincaré Map**.

2 In the **Settings** window for **Poincaré Map**, locate the **Data** section.

3 From the **Cut plane** list, choose **Cut Plane 1**.

4 In the **Ray Position Relative to Initial Position** toolbar, click **Plot**.

5 Click the **Zoom Extents** button in the **Graphics** toolbar.

Poincaré Map 2

1 Right-click **Poincaré Map 1** and choose **Duplicate**.

2 In the **Settings** window for **Poincaré Map**, locate the **Data** section.

3 From the **Cut plane** list, choose **Cut Plane 2**.

4 Locate the **Coloring and Style** section. From the **Color** list, choose **Blue**.

5 Select the **Radius scale factor** checkbox. In the associated text field, type 0.7.

6 Click to expand the **Title** section. From the **Title type** list, choose **None**.

7 In the **Ray Position Relative to Initial Position** toolbar, click **Plot**.

8 Click the **Zoom Extents** button in the **Graphics** toolbar. The plot should look like Figure 2.

The second method is to construct a **Phase Portrait** of the rays and verify that their positions and velocities are the same before and after passing through the cloak.

Change in Lateral Position

1 In the **Results** toolbar, click **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type Change in Lateral Position in the **Label** text field.

3 Locate the **Data** section. From the **Dataset** list, choose **Particle 1**.

4 From the **Time selection** list, choose **Last**.

5 Locate the **Plot Settings** section.

6 Select the **x-axis label** checkbox. In the associated text field, type `Initial position (mm)`.

7 Select the **y-axis label** checkbox. In the associated text field, type `Change in position (mm)`.

Particle 1

1 In the **Change in Lateral Position** toolbar, click **More Plots** and choose **Particle**.

2 In the **Settings** window for **Particle**, locate the **y-Axis Data** section.

3 In the **Expression** text field, type `qy-at(0,qy)`.

4 From the **Unit** list, choose **mm**.

5 Click to expand the **Title** section. From the **Title type** list, choose **None**.

6 Locate the **x-Axis Data** section. From the **Parameter** list, choose **Expression**.

7 In the **Expression** text field, type `at(0,qy)`.

8 From the **Unit** list, choose **mm**.

9 Click to expand the **Coloring and Style** section. Find the **Line markers** subsection. From the **Marker** list, choose **Point**.

10 In the **Change in Lateral Position** toolbar, click **Plot**. The plot should look like Figure 3. You can see that the rays do indeed return to their original position after passing through the cloak.