



Model created in COMSOL Multiphysics 6.4

# Time-Dependent Optimization

## *Introduction*

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Nonlinear systems that are driven by a sinusoidal excitation often evolve toward a periodic steady-state solution. Such systems occur in electromagnetics, plasma physics, and electrochemistry.

## *Model Definition*

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The test model solves the following ordinary differential equation:

$$\left(\frac{1}{f}\right)\frac{du}{dt} = a \sin^2(\omega t) - bu - cu^2 \quad (1)$$

where  $a$  is 0.25,  $b$  is 0.05, and  $c$  is 0.015. The frequency,  $f$ , is set to 1. Because the equation is nonlinear (due to the  $u^2$  term), it cannot be reformulated in the frequency domain by taking its Fourier transform. This ordinary differential equation is representative of the evolution of electronically excited metastable states in a capacitively coupled plasma. The initial value of  $u$  is set to be 0.25. For these conditions, it takes about 100 periods before  $u$  reaches its periodic steady-state solution. In a real plasma, it can take more than 100,000 RF cycles before the plasma has attained its periodic steady-state solution. Solving such a problem for so many RF cycles creates an insurmountable computational burden.

The periodic steady-state solution can be immediately computed using time-dependent optimization. A control variable,  $u_0$ , is used as the initial condition for [Equation 1](#). Next, an objective function is defined as:

$$g = (u - u_0)^2$$

When performing time-dependent optimization, the objective function is only evaluated at the last solution time. Thus, the global objective function seeks to make the initial value of  $u$  equal to the final value of  $u$  after exactly one period. This corresponds to the periodic steady-state solution to the problem.

## Results and Discussion

The time evolution of  $u$  is plotted in Figure 1. A close-up of the final few periods is plotted in Figure 2. This shows that  $u$  has indeed reached its periodic steady-state solution.

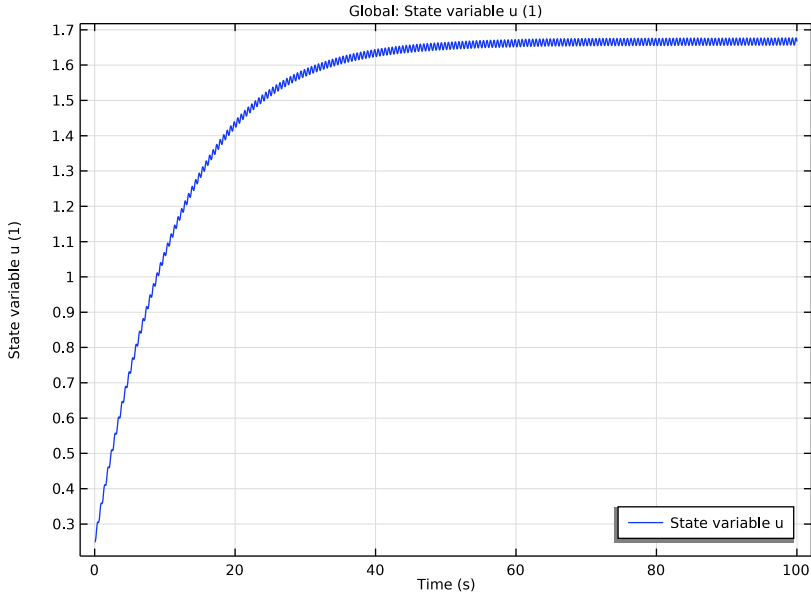
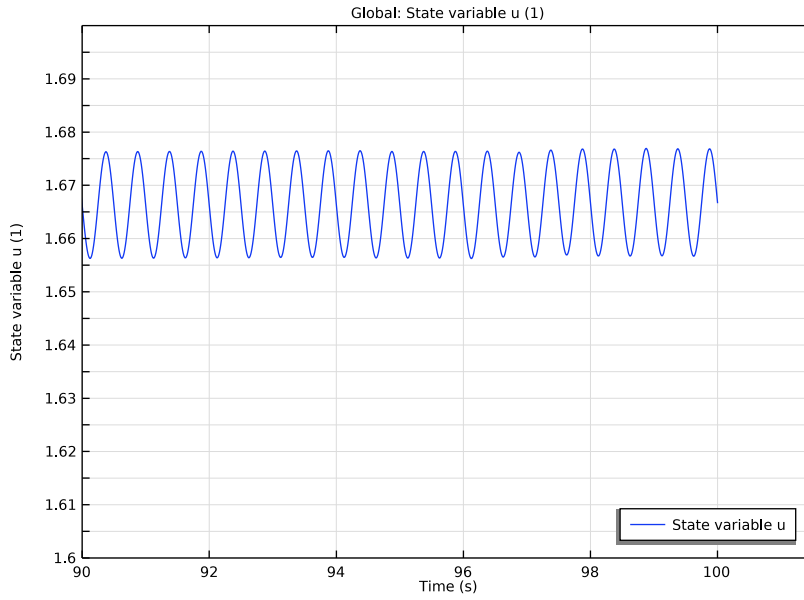


Figure 1: Plot of the evolution of  $u$  from its initial value of 0.25. There is a slow, steady increase in  $u$  over the first several periods along with oscillations at twice the driving frequency.

It is also obvious from Figure 2 that over 1 period, the value of  $u$  at the beginning of the period is the same as at the end of the period. In Figure 3 the solution computed by the optimization solver is shown. Note that the forward problem is only solved for 1 period. In total the optimization solver computes the solution to the forward problem only 6 times resulting in a much reduced simulation time.



*Figure 2: Close up of the last several cycles of the forward problem. The model has clearly reached its periodic steady state solution after 100 cycles.*

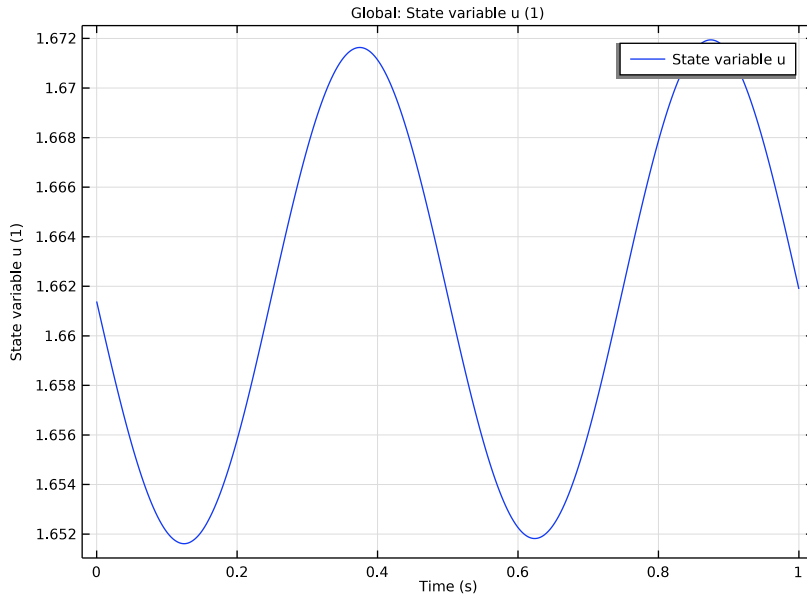


Figure 3: Plot of the solution computed by the optimization solver.

### Reference

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1. D.P. Lymberopoulos and D.J. Economou, “Fluid simulation of glow discharges: Effect of metastable atoms in argon,” *J. Appl. Phys.* vol. 73, no. 8, 1993.

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**Application Library path:** Optimization\_Module/Parameter\_Estimation/  
time\_dependent\_optimization

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### Modeling Instructions

From the **File** menu, choose **New**.

#### NEW

In the **New** window, click  **Model Wizard**.

## MODEL WIZARD

- 1 In the **Model Wizard** window, click  **OD**.
- 2 In the **Select Physics** tree, select **Mathematics > ODE and DAE Interfaces > Global ODEs and DAEs (ge)**.
- 3 Click **Add**.
- 4 Click  **Study**.
- 5 In the **Select Study** tree, select **General Studies > Time Dependent**.
- 6 Click  **Done**.

## GLOBAL DEFINITIONS

### *Parameters I*

- 1 In the **Model Builder** window, under **Global Definitions** click **Parameters I**.
- 2 In the **Settings** window for **Parameters**, locate the **Parameters** section.
- 3 In the table, enter the following settings:

Name	Expression	Value	Description
a	0.25	0.25	ODE constant 1
b	0.05	0.05	ODE constant 2
c	0.015	0.015	ODE constant 3
u0	0.25	0.25	Initial value
f	1[Hz]	1 Hz	Frequency
w	2*pi*f	6.2832 Hz	Angular frequency

## GLOBAL ODES AND DAES (GE)

Define the ordinary differential equation with the periodic forcing function.

### *Global Equations I (ODEI)*

- 1 In the **Model Builder** window, under **Component I (comp1) > Global ODEs and DAEs (ge)** click **Global Equations I (ODEI)**.
- 2 In the **Settings** window for **Global Equations**, locate the **Global Equations** section.
- 3 In the table, enter the following settings:

Name	$f(u, ut, utt, t)$ (I)	Initial value (u_0) (I)	Initial value (ut_0) (I/s)
u	$((1/f)*ut - a*\sin(w*t)^2 + b*u + c*u^2)$	u0	0



## STUDY 1

The model needs to be solved for 100 periods before it reaches its periodic steady state solution.

### *Step 1: Time Dependent*


- 1 In the **Model Builder** window, under **Study 1** click **Step 1: Time Dependent**.
- 2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.
- 3 From the **Tolerance** list, choose **User controlled**.
- 4 In the **Relative tolerance** text field, type  $1e-5$ .
- 5 In the **Output times** text field, type range (0,0.01,100).

### *Solution 1 (sol1)*


- 1 In the **Study** toolbar, click  **Show Default Solver**.
- 2 In the **Model Builder** window, expand the **Solution 1 (sol1)** node, then click **Time-Dependent Solver 1**.
- 3 In the **Settings** window for **Time-Dependent Solver**, click to expand the **Absolute Tolerance** section.
- 4 From the **Tolerance method** list, choose **Manual**.
- 5 In the **Absolute tolerance** text field, type  $0.0001$ .
- 6 In the **Study** toolbar, click  **Compute**.

## RESULTS

### *ID Plot Group 1*

- 1 In the **Settings** window for **ID Plot Group**, locate the **Legend** section.
- 2 From the **Position** list, choose **Lower right**.
- 3 In the **ID Plot Group 1** toolbar, click  **Plot**.



### *ID Plot Group 2*

- 1 Right-click **Results > ID Plot Group 1** and choose **Duplicate**.
- 2 In the **Settings** window for **ID Plot Group**, locate the **Axis** section.
- 3 Select the **Manual axis limits** checkbox.
- 4 In the **x minimum** text field, type 90.
- 5 In the **y minimum** text field, type 1.6.
- 6 In the **y maximum** text field, type 1.7.
- 7 In the **ID Plot Group 2** toolbar, click  **Plot**.

## ROOT

Now add another study with an **Optimization** step which can be used to immediately compute the periodic steady state solution for the differential equation.

## ADD STUDY


- 1 In the **Home** toolbar, click  **Add Study** to open the **Add Study** window.
- 2 Go to the **Add Study** window.
- 3 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies** > **Time Dependent**.
- 4 Click the **Add Study** button in the window toolbar.
- 5 In the **Home** toolbar, click  **Add Study** to close the **Add Study** window.

## STUDY 2

### *Step 1: Time Dependent*

- 1 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.
- 2 In the **Output times** text field, type range (0,0.002,1).
- 3 From the **Tolerance** list, choose **User controlled**.
- 4 In the **Relative tolerance** text field, type 1e-5.

### *General Optimization*

- 1 In the **Study** toolbar, click  **Optimization** and choose **General Optimization**.
- 2 In the **Settings** window for **General Optimization**, locate the **Optimization Solver** section.
- 3 From the **Method** list, choose **IPOPT**.  
Add the difference between initial and final value in a cycle as error measure to be minimized.
- 4 Locate the **Objective Function** section. In the table, enter the following settings:

Expression	Description	Evaluate for
(comp1.u-u0)^2	Squared error	Time Dependent



Next, add the initial value as control parameter and set suitable bounds to help the solver.

- 5 Locate the **Control Variables and Parameters** section. Click  **Add**.

6 In the table, enter the following settings:

Parameter	Initial value	Scale	Lower bound	Upper bound	Unit
u0 (Initial value)	0.25	1	0	5	


*Solution 2 (sol2)*

- 1 In the **Study** toolbar, click  **Show Default Solver**.
- 2 In the **Model Builder** window, expand the **Solution 2 (sol2)** node.
- 3 In the **Model Builder** window, expand the **Study 2 > Solver Configurations > Solution 2 (sol2) > Optimization Solver 1** node, then click **Time-Dependent Solver 1**.
- 4 In the **Settings** window for **Time-Dependent Solver**, click to expand the **Absolute Tolerance** section.
- 5 From the **Tolerance method** list, choose **Manual**.
- 6 In the **Absolute tolerance** text field, type  $1e-5$ .
- 7 In the **Study** toolbar, click  **Compute**.

The solver will issue a warning as a reminder that the objective function is only evaluated at the final time — which is indeed the desired behavior in this model.

## RESULTS

*1D Plot Group 3*

- 1 Click the  **Zoom Extents** button in the **Graphics** toolbar.

The periodic steady state solution is obtained (compare to [Figure 2](#)).