Created in COMSOL Multiphysics 6.0

Uncertainty Quantification of the Ishigami Function

Introduction

This example demonstrates how to perform uncertainty quantification analysis of the Ishigami function. This random function of three variables is a well-known benchmark used to test global sensitivity analysis and uncertainty quantification algorithms. The mean, standard deviation, maximum, and mininum values as well as Sobol indices of the Ishigami function can be calculated analytically for the input distributions used here.

For this test problem, the Ishigami function is

$$f(X_1, X_2, X_3) = \sin(X_1) + a(\sin(X_2))^2 + bX_3^4\sin(X_1)$$

where X_1, X_2, and X_3 are independent uniformly distributed random variables in $[-\pi,+\pi]$ with $a = 7$ and $b = 0.1$.

The function can be visualized in 3D by using, for example, a slice plot as in Figure 1.

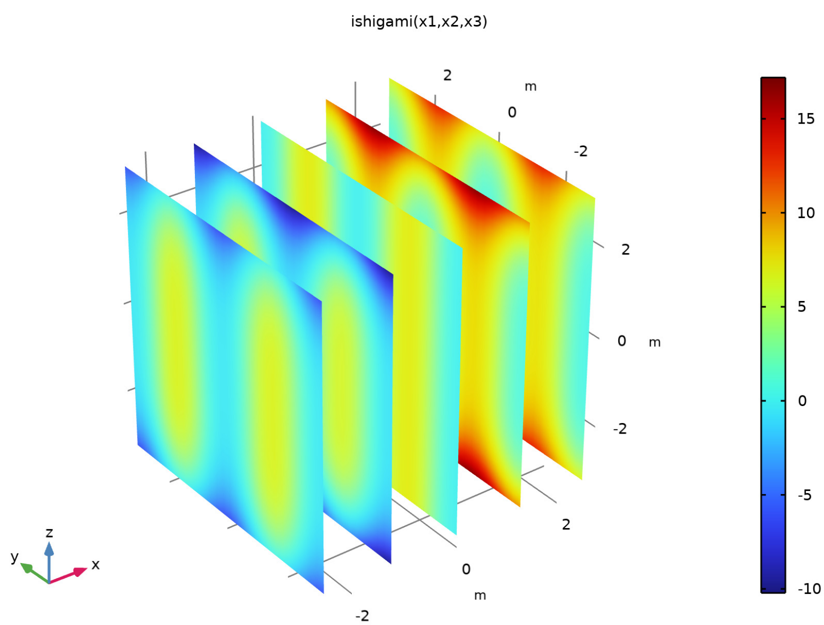


Figure 1: Slice plot of the Ishigami function.

The analytically computed values are according to Table 1.

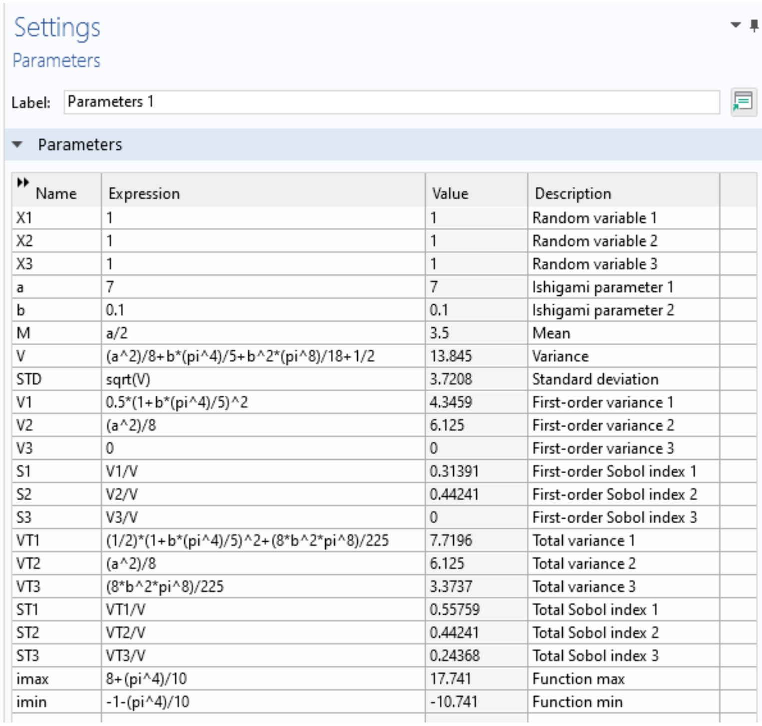
TABLE 1: ANALYTICAL BENCHMARK VALUES.

QUANTITY	EXPRESSION	NUMERICAL VALUE (ROUNDED)
Mean value	`a/2`	`3.5`
Variance (V)	`(a^2)/8+b*(pi^4)/5+b^2*(pi^8)/18+1/2`	`13.845`
Maximum	`8+(pi^4)/10`	`17.741`
Minimum	`-1-(pi^4)/10`	`-10.741`
Standard deviation	`sqrt(V)`	`3.7208`
First-order Sobol index X_1	`(0.5*(1+b*(pi^4)/5)^2)/V`	`0.31391`
First-order Sobol index X_2	`((a^2)/8)/V`	`0.44241`
First-order Sobol index X_3	`0`	`0`
Total Sobol index X_1	`((1/2)*(1+b*(pi^4)/5)^2+(8*b^2*pi^8)/225)/V`	`0.55759`
Total Sobol index X_2	`((a^2)/8/V`	`0.44241`
Total Sobol index X_3	`((8*b^2*pi^8)/225/V`	`0.24368`

For reference, these values are entered as global parameters in the model.

Model Definition

The model runs through 3 uncertainty quantification studies: **Screening**, **Sensitivity analysis**, and **Uncertainty Propagation** using the Ishigami function as the quantity of interest. In order to perform the uncertainty quantification analysis, the three random variables need to be defined as global parameters using arbitrary values. The actual values for these variables will, during the simulation, be randomized by the uncertainty quantification algorithms. All the global parameters in the model are shown in Figure 2.



Name	Expression	Value	Description
X1	1	1	Random variable 1
X2	1	1	Random variable 2
X3	1	1	Random variable 3
a	7	7	Ishigami parameter 1
b	0.1	0.1	Ishigami parameter 2
M	a/2	3.5	Mean
V	(a^2)/8+b*(pi^4)/5+b^2*(pi^8)/18+1/2	13.845	Variance
STD	sqrt(V)	3.7208	Standard deviation
V1	0.5*(1+b*(pi^4)/5)^2	4.3459	First-order variance 1
V2	(a^2)/8	6.125	First-order variance 2
V3	0	0	First-order variance 3
S1	V1/V	0.31391	First-order Sobol index 1
S2	V2/V	0.44241	First-order Sobol index 2
S3	V3/V	0	First-order Sobol index 3
VT1	(1/2)*(1+b*(pi^4)/5)^2+(8*b^2*pi^8)/225	7.7196	Total variance 1
VT2	(a^2)/8	6.125	Total variance 2
VT3	(8*b^2*pi^8)/225	3.3737	Total variance 3
ST1	VT1/V	0.55759	Total Sobol index 1
ST2	VT2/V	0.44241	Total Sobol index 2
ST3	VT3/V	0.24368	Total Sobol index 3
imax	8+(pi^4)/10	17.741	Function max
imin	-1-(pi^4)/10	-10.741	Function min

Figure 2: The model parameters.

The Ishigami function is defined as an analytic function with three input arguments as shown in Figure 3.

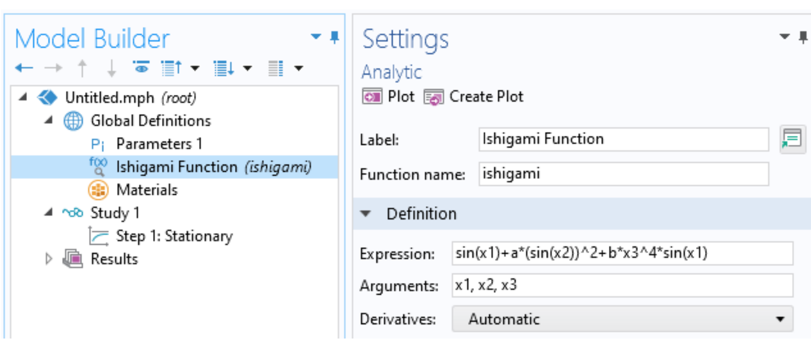


Figure 3: The Ishigami function entered as an Analytic function, `ishigami`.

Results and Discussion

The sensitivity analysis shows that the computed Sobol indices are consistent with the true analytical values, as shown in Figure 4 below.

	ishigami(X1,X2,X3), First order	ishigami(X1,X2,X3), Total
X1	0.31363	0.55131
X2	0.44844	0.44870
X3	8.2735E-5	0.23785

Figure 4: The computed Sobol indices.

Similarly, the values for mean, standard deviation (STD), minimum, and maximum are consistent with the analytical values, as shown in Figure 5.

	Mean	STD	Minimum	Maximum	Lower 90%	Upper 90%	Lower 95%	Upper 95%	Lower 99%	Upper 99%
ishigami(X1,X2,X3)	3.5186	3.6494	-10.094	17.337	-2.0346	9.2073	-3.6171	10.986	-6.8413	13.770

Figure 5: The computed values for mean, standard deviation, minimum, maximum, and confidence intervals.

The accuracy of the results can be increased by lowering tolerances or increasing the number of sample input points.

The computed kernel density estimation is displayed in Figure 6.

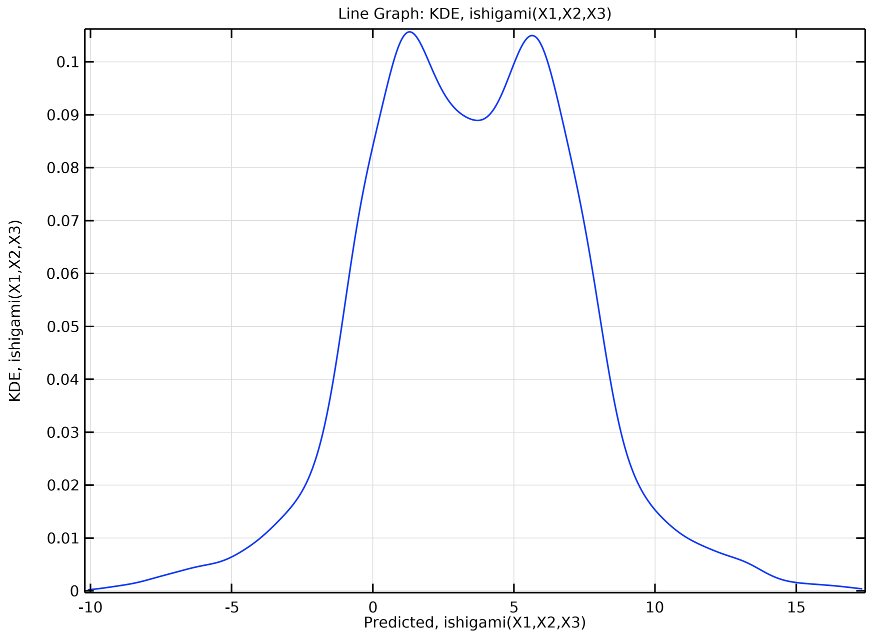


Figure 6: The KDE plot for the Ishigami function.

These uncertainty quantification results can be compared not only with the analytical values but also with that of the direct Monte Carlo simulation performed in the model Direct Monte Carlo Simulation of the Ishigami Function.

Reference

1. T. Ishigami and T. Homma, "An importance quantification technique in uncertainty analysis for computer models," *Proc. First Int'l Symp. Uncertainty Modeling and Analysis*, IEEE, pp. 398-403, 1990.

Application Library path: `Uncertainty_Quantification_Module/Tutorials/ishigami_function_uncertainty_quantification`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Blank Model**.

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Studies** subsection. In the **Select Study** tree, select **Preset Studies for Selected Physics Interfaces>Stationary**.

4 Click **Add Study** in the window toolbar.

5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

GLOBAL DEFINITIONS

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

Name	Expression	Value	Description
X1	1	1	Random variable 1
X2	1	1	Random variable 2
X3	1	1	Random variable 3
a	7	7	Ishigami parameter 1
b	0.1	0.1	Ishigami parameter 2
M	a/2	3.5	Mean
V	(a^2)/8+b*(pi^4)/5+ b^2*(pi^8)/18+1/2	13.845	Variance
STD	sqrt(V)	3.7208	Standard deviation
V1	0.5*(1+b*(pi^4)/5)^2	4.3459	First-order variance 1
V2	(a^2)/8	6.125	First-order variance 2
V3	0	0	First-order variance 3
S1	V1/V	0.31391	First-order Sobol index 1

Name	Expression	Value	Description
S2	V2/V	0.44241	First-order Sobol index 2
S3	V3/V	0	First-order Sobol index 3
VT1	(1/2)*(1+b*(pi^4)/5)^2+(8*b^2*pi^8)/225	7.7196	Total variance 1
VT2	(a^2)/8	6.125	Total variance 2
VT3	(8*b^2*pi^8)/225	3.3737	Total variance 3
ST1	VT1/V	0.55759	Total Sobol index 1
ST2	VT2/V	0.44241	Total Sobol index 2
ST3	VT3/V	0.24368	Total Sobol index 3
imax	8+(pi^4)/10	17.741	Function max
imin	-1-(pi^4)/10	-10.741	Function min

Ishigami Function

1 In the **Home** toolbar, click f(x) **Functions** and choose **Global>Analytic**.

2 In the **Settings** window for **Analytic**, type `ishigami` in the **Function name** text field.

3 Locate the **Definition** section. In the **Expression** text field, type `sin(x1)+a*(sin(x2))^2+b*x3^4*sin(x1)`.

4 In the **Arguments** text field, type `x1,x2,x3`.

5 In the **Label** text field, type `Ishigami Function`.

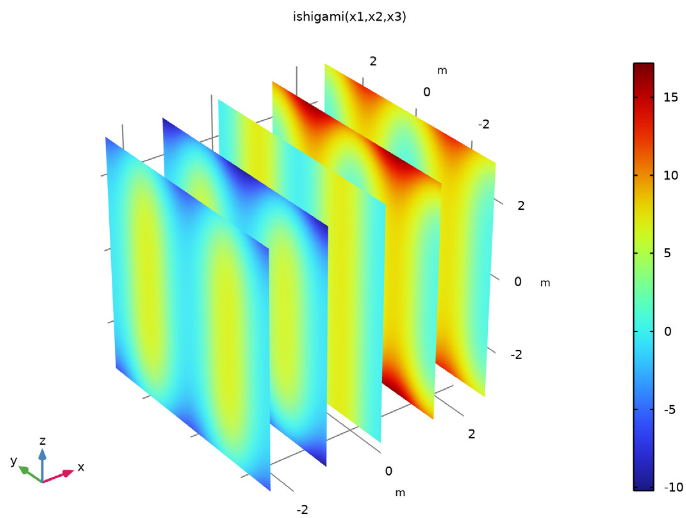
6 Locate the **Plot Parameters** section. In the table, enter the following settings:

Argument	Lower limit	Upper limit	Unit
x1	-pi	pi	
x2	-pi	pi	
x3	-pi	pi	

7 Click **Create Plot**.

RESULTS

3D Plot Group 1



STUDY 1

Uncertainty Quantification

1 In the **Model Builder** window, right-click **Study 1** and choose **Uncertainty Quantification> Uncertainty Quantification**.

2 In the **Settings** window for **Uncertainty Quantification**, locate the **Quantities of Interest** section.

3 Click **Add**.

4 In the table, enter the following settings:

Expression	Description	Individual solution to use
`ishigami(X1,X2,X3)`	`Ishigami Function`	From "Solution to use"

5 Locate the **Input Parameters** section. Find the **Input parameters table** subsection. Click **Add** three times.

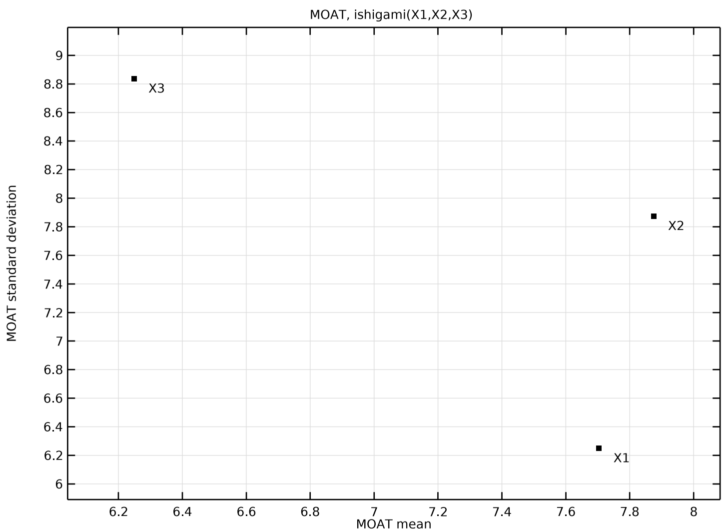
6 In the table, enter the following settings:

Parameter	Distribution	Lower bound	Upper bound
X1 (Random variable 1)	Uniform	`-pi`	`pi`
X2 (Random variable 2)	Uniform	`-pi`	`pi`
X3 (Random variable 3)	Uniform	`-pi`	`pi`

7 In the **Home** toolbar, click **Compute**.

RESULTS

MOAT, ishigami(X1,X2,X3)



The Screening study shows that all parameters are influential and that the parameter `X3` has a nonlinear influence on the Ishigami function, or that it is interacting with the other input parameters, or both.

STUDY 1

Uncertainty Quantification

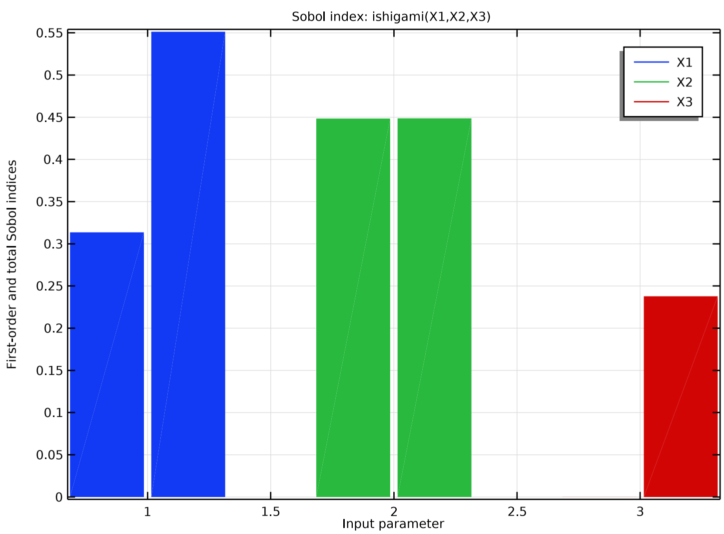
In the **Model Builder** window, under **Study 1** right-click **Uncertainty Quantification** and choose **Add New Uncertainty Quantification Study For>Sensitivity Analysis**.

STUDY 2

Uncertainty Quantification

To achive a high level of accuracy, change from the default **Compute type**, which is **Improve and analyze**, to **Compute and analyze**. This option will not reuse any results from previous model evaluations but instead start from scratch.

1 In the **Model Builder** window, under **Study 2** click **Uncertainty Quantification**.
2 In the **Settings** window for **Uncertainty Quantification**, locate the **Uncertainty Quantification Settings** section.
3 From the **Compute action** list, choose **Compute and analyze**.
4 In the **Home** toolbar, click **Compute**.



The Sensitivity analysis study computes Sobol indices that are consistent with the analytical values.

5 Right-click **Study 2>Uncertainty Quantification** and choose **Add New Uncertainty Quantification Study For>Uncertainty Propagation**.

STUDY 3

Uncertainty Quantification

Now, change the **Surrogate model** to **Adaptive sparse polynomial chaos expansion**. For the Ishigami function, the polynomical chaos expansion surrogate model turns out to be much more efficient than the default **Adaptive Gaussian process** option.

1 In the **Model Builder** window, under **Study 3** click **Uncertainty Quantification**.

2 In the **Settings** window for **Uncertainty Quantification**, locate the **Uncertainty Quantification Settings** section.

3 Find the **Surrogate model settings** subsection. From the **Surrogate model** list, choose **Adaptive sparse polynomial chaos expansion**.

Again, to achive a high level of accuracy, change to **Compute and analyze**. This option will not reuse any results from previous model evaluations but instead start from scratch.

4 From the **Compute action** list, choose **Compute and analyze**.

5 In the **Home** toolbar, click **Compute**.