Created in COMSOL Multiphysics 6.0

# Direct Monte Carlo Simulation of the Ishigami Function

## *Introduction*

This example demonstrates how to perform a direct Monte Carlo simulation of the Ishigami function. This random function of three variables is a well-known benchmark used to test global sensitivity analysis and uncertainty quantification algorithms. The mean, standard deviation, maximum, and mininum values of the Ishigami function can be calculated analytically for the input distributions used here.

For this test problem, the Ishigami function is

$$f(X_1, X_2, X_3) = \sin(X_1) + a(\sin(X_2))^2 + bX_3^4\sin(X_1)$$

where $X_1$, $X_2$, and $X_3$ are independent uniformly distributed random variables in $[-\pi,+\pi]$ with $a = 7$ and $b = 0.1$.

The analytically computed values are according to Table 1.

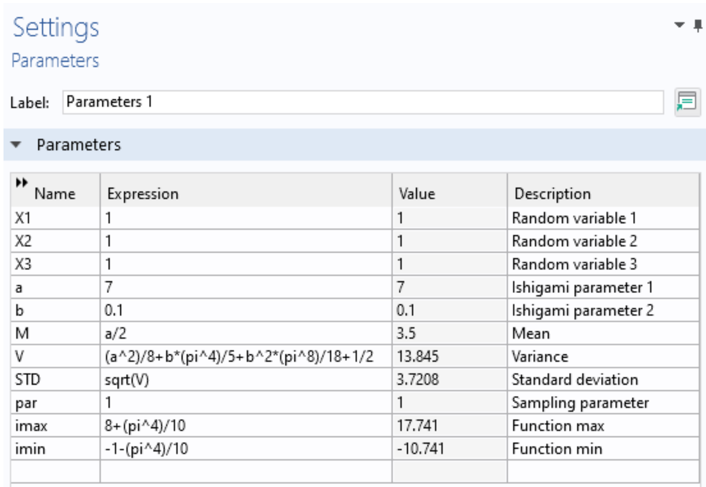
TABLE 1: ANALYTICAL BENCHMARK VALUES.

| Quantity | Expression | Numerical value (rounded) |
|---|---|---|
| Mean value | `a/2` | `3.5` |
| Variance | `(a^2)/8+b*(pi^4)/5+b^2*(pi^8)/18+1/2` | `13.845` |
| Maximum | `8+(pi^4)/10` | `17.741` |
| Minimum | `-1-(pi^4)/10` | `-10.741` |
| Standard deviation | `sqrt(V)` | `3.7208` |

For reference, these values are entered as global parameters in the model.

## *Model Definition*

In order to perform a direct Monte Carlo simulation, the three random variables need to be defined as global parameters using arbitrary values. The actual values for these variables during the simulation will be randomized. Figure 1 shows all global parameters in the model.



| Name | Expression | Value | Description |
|---|---|---|---|
| X1 | 1 | 1 | Random variable 1 |
| X2 | 1 | 1 | Random variable 2 |
| X3 | 1 | 1 | Random variable 3 |
| a | 7 | 7 | Ishigami parameter 1 |
| b | 0.1 | 0.1 | Ishigami parameter 2 |
| M | a/2 | 3.5 | Mean |
| V | (a^2)/8+b*(pi^4)/5+b^2*(pi^8)/18+1/2 | 13.845 | Variance |
| STD | sqrt(V) | 3.7208 | Standard deviation |
| par | 1 | 1 | Sampling parameter |
| imax | 8+(pi^4)/10 | 17.741 | Function max |
| imin | -1-(pi^4)/10 | -10.741 | Function min |

*Figure 1: The model parameters.*

The Sampling parameter `par` is used to generate vector of random values for the random variables `X1`, `X2`, and `X3`.

The Ishigami function is defined as an analytic function with three input arguments.

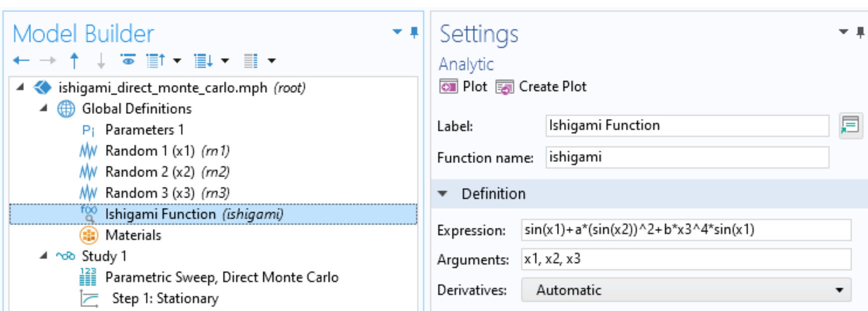



*Figure 2: The Ishigami function entered as an Analytic function,* `ishigami`.

Three uniform random functions — `rn1`, `rn2`, and `rn2` — are defined and later used in the Monte Carlo simulation. Each of the random variables has a unique random seed to ensure that they emulate independent random variables.

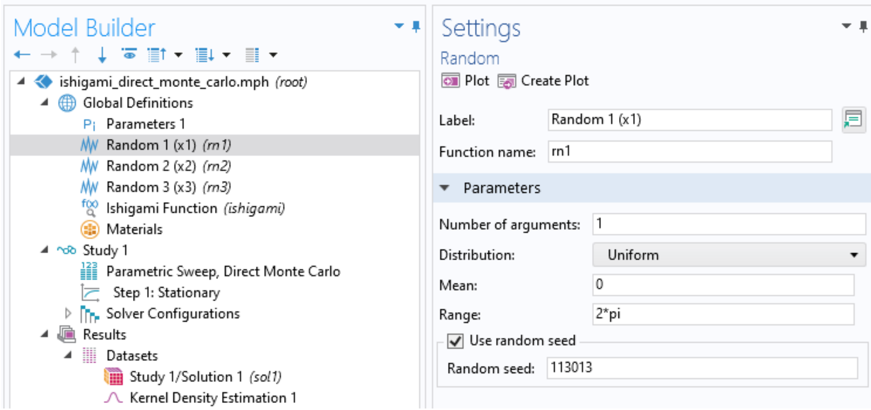



*Figure 3: The random functions used for the Monte Carlo simulation.*

The main part of the simulation is a **Parametric Sweep** and a **Stationary** study. The **Stationary** study is necessary to generate all function values used for the results processing. In the model, 10,000 sample points are generated for each of the random variables. This is achieved by running a parametric sweep between 1 and 10,000 in steps of 1. To increase the accuracy of the simulation you can easily increase this value by 10 or 100 times.

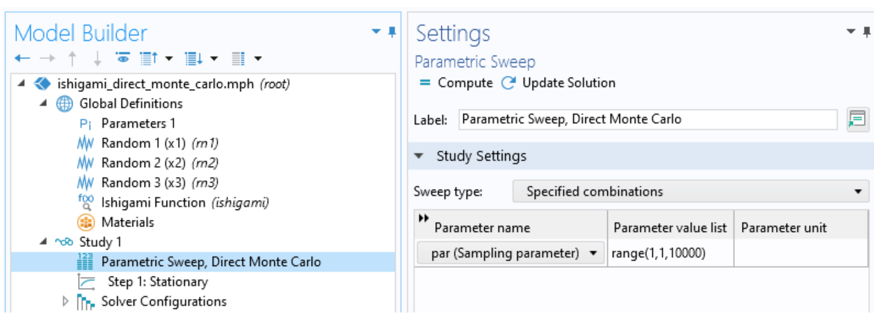



*Figure 4: The Parametric Sweep definition used for the Monte Carlo simulation.*

The Monte Carlo simulation is performed by means of an **Evaluation Group** that outputs the simulated function values to a table.

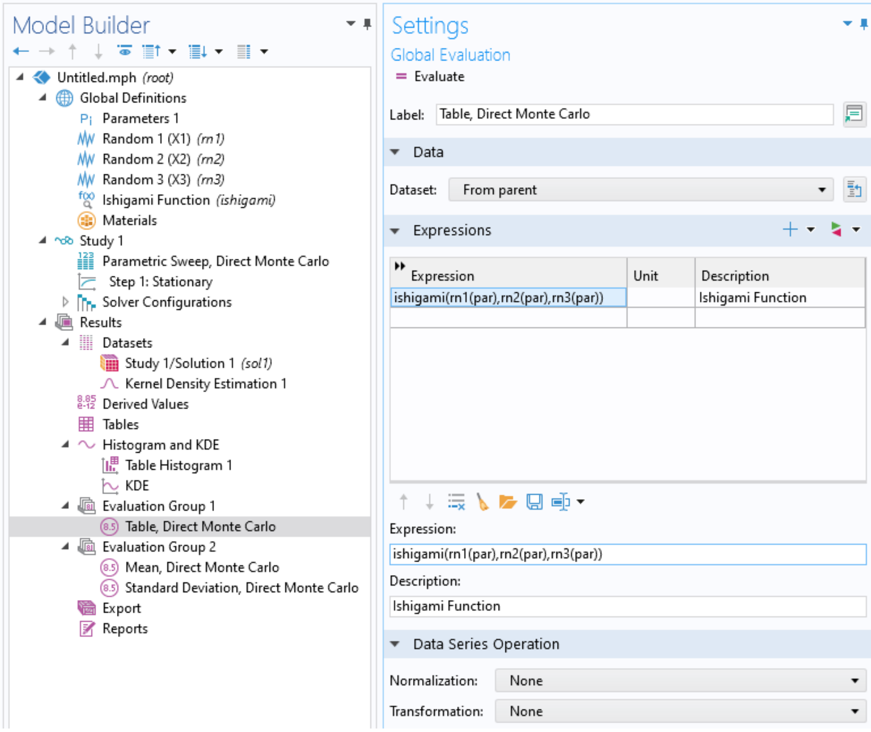



*Figure 5: The Evaluation Group used to evaluate the Ishigami function.*

The results compares a **Table Histogram** plot with a **KDE** plot (kernel density estimation).

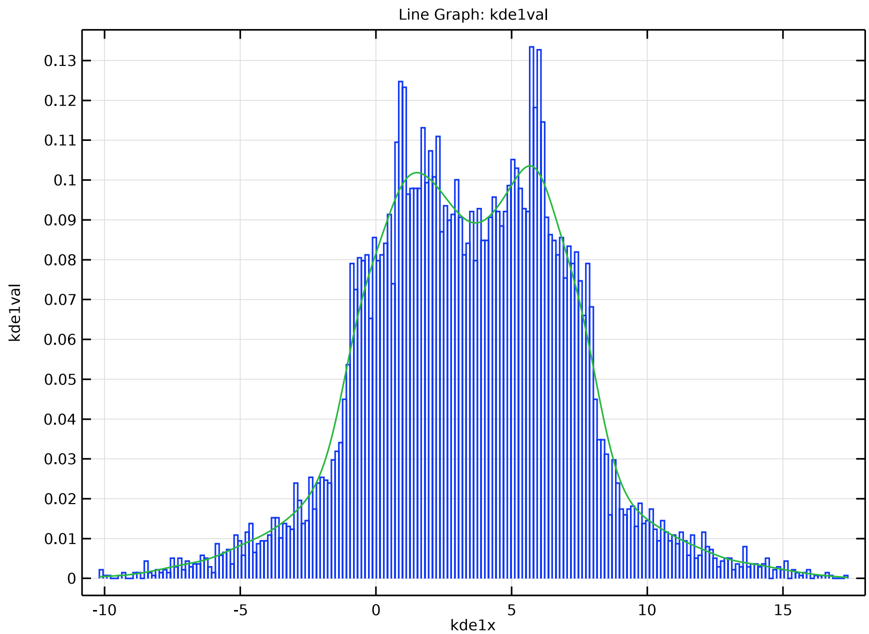
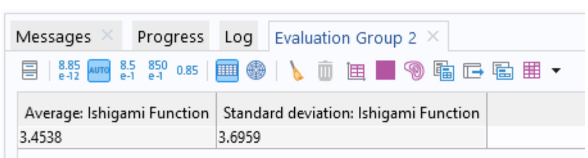



*Figure 6: A Table Histogram plot and KDE plot.*

The mean and standard deviation are computed as **Data Series Operations** on the **Evaluation Group** table.




| Average: Ishigami Function | Standard deviation: Ishigami Function |
|---|---|
| 3.4538 | 3.6959 |

*Figure 7: The computed mean and standard deviation.*

These values should be compared with the analytical values of the mean and standard deviation of `3.5` and `3.7208`, respectively. To get higher accuracy, increase the number of steps in the sweep.

## *Reference*

1. T. Ishigami and T. Homma, "An importance quantification technique in uncertainty analysis for computer models," *Proc. First Int'l Symp. Uncertainty Modeling and Analysis*, IEEE, pp. 398-403, 1990.

**Application Library path:** `Uncertainty_Quantification_Module/Tutorials/ishigami_function_direct_monte_carlo`

## *Modeling Instructions*

From the **File** menu, choose **New**.

### NEW

In the **New** window, click **Blank Model**.

### ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Studies** subsection. In the **Select Study** tree, select **Preset Studies for Selected Physics Interfaces>Stationary**.

4 Click **Add Study** in the window toolbar.

5 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

### GLOBAL DEFINITIONS

*Parameters 1*

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.

2 In the **Settings** window for **Parameters**, locate the **Parameters** section.

3 In the table, enter the following settings:

| Name | Expression | Value | Description |
|---|---|---|---|
| X1 | 1 | 1 | Random variable 1 |
| X2 | 1 | 1 | Random variable 2 |
| X3 | 1 | 1 | Random variable 3 |
| a | 7 | 7 | Ishigami parameter 1 |

| Name | Expression | Value | Description |
|---|---|---|---|
| b | 0.1 | 0.1 | Ishigami parameter 2 |
| M | a/2 | 3.5 | Mean |
| V | (a^2)/8+b*(pi^4)/5+ b^2*(pi^8)/18+1/2 | 13.845 | Variance |
| STD | sqrt(V) | 3.7208 | Standard deviation |
| par | 1 | 1 | Sampling parameter |
| imax | 8+(pi^4)/10 | 17.741 | Function max |
| imin | -1-(pi^4)/10 | -10.741 | Function min |

*Random 1 (X1)*

1 In the **Home** toolbar, click f(x) **Functions** and choose **Global>Random**.

2 In the **Settings** window for **Random**, type `Random 1 (X1)` in the **Label** text field.

3 Locate the **Parameters** section. In the **Range** text field, type `2*pi`.

4 Select the **Use random seed** check box.

*Random 2 (X2)*

1 In the **Home** toolbar, click f(x) **Functions** and choose **Global>Random**.

2 In the **Settings** window for **Random**, type `Random 2 (X2)` in the **Label** text field.

3 Locate the **Parameters** section. In the **Range** text field, type `2*pi`.

4 Select the **Use random seed** check box.

*Random 3 (X3)*

1 In the **Home** toolbar, click f(x) **Functions** and choose **Global>Random**.

2 In the **Settings** window for **Random**, type `Random 3 (X3)` in the **Label** text field.

3 Locate the **Parameters** section. In the **Range** text field, type `2*pi`.

4 Select the **Use random seed** check box.

*Ishigami Function*

1 In the **Home** toolbar, click f(x) **Functions** and choose **Global>Analytic**.

2 In the **Settings** window for **Analytic**, type `ishigami` in the **Function name** text field.

3 Locate the **Definition** section. In the **Expression** text field, type `sin(x1)+a*(sin(x2))^2+b*x3^4*sin(x1)`.

4 In the **Arguments** text field, type `x1,x2,x3`.

5 In the **Label** text field, type `Ishigami Function`.

## STUDY 1

*Parametric Sweep, Direct Monte Carlo*

1 In the **Study** toolbar, click **Parametric Sweep**.

2 In the **Settings** window for **Parametric Sweep**, type `Parametric Sweep, Direct Monte Carlo` in the **Label** text field.

3 Locate the **Study Settings** section. Click **Add**.

4 In the table, enter the following settings:

| Parameter name | Parameter value list | Parameter unit |
|---|---|---|
| par (Sampling parameter) | `range(1,1,10000)` | |

5 In the **Study** toolbar, click **Compute**.

## RESULTS

*Evaluation Group 1*

1 In the **Model Builder** window, expand the **Results** node.

2 Right-click **Results** and choose **Evaluation Group**.

*Table, Direct Monte Carlo*

1 In the **Model Builder** window, right-click **Evaluation Group 1** and choose **Global Evaluation**.

2 In the **Settings** window for **Global Evaluation**, locate the **Expressions** section.

3 In the table, enter the following settings:

| Expression | Unit | Description |
|---|---|---|
| `ishigami(rn1(par),rn2(par),rn3(par))` | | `Ishigami Function` |

4 In the **Evaluation Group 1** toolbar, click **Evaluate**.

5 In the **Label** text field, type `Table, Direct Monte Carlo`.

*Evaluation Group 2*

In the **Results** toolbar, click **Evaluation Group**.

*Mean, Direct Monte Carlo*

1 Right-click **Evaluation Group 2** and choose **Global Evaluation**.

2 In the **Settings** window for **Global Evaluation**, type `Mean, Direct Monte Carlo` in the **Label** text field.

**3** Locate the **Expressions** section. In the table, enter the following settings:

| Expression | Unit | Description |
|---|---|---|
| `ishigami(rn1(par),rn2(par),rn3(par))` | | `Ishigami Function` |

**4** Locate the **Data Series Operation** section. From the **Transformation** list, choose **Average**.

*Standard Deviation, Direct Monte Carlo*

**1** In the **Model Builder** window, right-click **Evaluation Group 2** and choose **Global Evaluation**.

**2** In the **Settings** window for **Global Evaluation**, type `Standard Deviation, Direct Monte Carlo` in the **Label** text field.

**3** Locate the **Expressions** section. In the table, enter the following settings:

| Expression | Unit | Description |
|---|---|---|
| `ishigami(rn1(par),rn2(par),rn3(par))` | | `Ishigami Function` |

**4** Locate the **Data Series Operation** section. From the **Transformation** list, choose **Standard deviation**.

*Evaluation Group 2*

**1** In the **Model Builder** window, click **Evaluation Group 2**.

**2** In the **Evaluation Group 2** toolbar, click **Evaluate**.

*Kernel Density Estimation 1*

**1** In the **Model Builder** window, expand the **Results>Datasets** node.

**2** Right-click **Results>Datasets** and choose **More Datasets>Kernel Density Estimation**.

**3** In the **Settings** window for **Kernel Density Estimation**, locate the **Data** section.

**4** From the **Source** list, choose **Evaluation group**.

**5** Find the **Columns** subsection. From the **x-coordinates** list, choose **Ishigami Function**.

**6** Click to expand the **Advanced** section.

*Histogram and KDE*

**1** In the **Results** toolbar, click **1D Plot Group**.

**2** In the **Settings** window for **1D Plot Group**, type `Histogram and KDE` in the **Label** text field.

*Table Histogram 1*

**1** In the **Histogram and KDE** toolbar, click **More Plots** and choose **Table Histogram**.

**2** In the **Settings** window for **Table Histogram**, locate the **Data** section.

3 From the **Source** list, choose **Evaluation group**.

4 From the **x-coordinates** list, choose **Ishigami Function**.

5 Locate the **Bins** section. In the **Number** text field, type `200`.

6 Locate the **Output** section. From the **Normalization** list, choose **Integral**.

7 Click to expand the **Coloring and Style** section.

*KDE*

1 In the **Model Builder** window, right-click **Histogram and KDE** and choose **Line Graph**.

2 In the **Settings** window for **Line Graph**, type `KDE` in the **Label** text field.

3 Locate the **y-Axis Data** section. In the **Expression** text field, type `kde1val`.

4 Locate the **Data** section. From the **Dataset** list, choose **Kernel Density Estimation 1**.

5 In the **Histogram and KDE** toolbar, click **Plot**.

6 Click the **Zoom Extents** button in the **Graphics** toolbar.