

# Ion Cyclotron Motion

# Introduction

If a charged particle is placed in a uniform magnetic field, it moves in a helical pattern about a fixed gyro radius. The gyro radius, which is also known as the Larmor or cyclotron radius, is given by the simple equation:

$$r_{\rm L} = \frac{mv_{\perp}}{ZeB}$$

- $r_{\rm L}$  (SI unit: m) is the Larmor radius,
- $v_{\perp}$  (SI unit: m/s) is the velocity component orthogonal to the magnetic field,
- Z (dimensionless) is the particle charge number,
- $e = 1.602176634 \times 10^{-19}$  C is the elementary charge,
- *m* (SI unit: kg) is the particle mass, and
- **B** (SI unit: T) is the magnitude of the magnetic flux density.

This model computes the trajectory of an ion in a uniform magnetic field using the Newtonian, Lagrangian, and Hamiltonian formulations available in the Mathematical Particle Tracing interface.

# Model Definition

The equations of motion for a charge in a magnetic field can be determined from the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \mathbf{v}} \right) = \frac{\partial L}{\partial \mathbf{q}} \tag{1}$$

where  $\mathbf{v}$  is the particle velocity,  $\mathbf{q}$  is the particle position, and L (SI unit: J) is the Lagrangian, which is defined as:

$$L = \frac{m(\mathbf{v} \cdot \mathbf{v})}{2} + q(\mathbf{v} \cdot \mathbf{A})$$

This form of the Lagrangian is valid for nonrelativistic particles; that is, the particle velocity is much less than the speed of light. The contribution due to the electric potential is neglected. The Hamiltonian is related to the Lagrangian via:

$$H = \mathbf{v} \cdot \frac{\partial L}{\partial \mathbf{v}} - L$$

Introducing the generalized momentum of the particle,  $\mathbf{P}$  (SI unit: kg·m/s), the Hamiltonian becomes:

$$H = \frac{\left(\mathbf{P} - q\mathbf{A}\right)^2}{2m}$$

In order to derive the equations of motion for the Newtonian formulation, start with the right-hand side of Equation 1:

$$\frac{\partial L}{\partial \mathbf{q}} = \nabla L = q \nabla (\mathbf{A} \cdot \mathbf{v}) = q (\mathbf{v} \cdot \nabla) \mathbf{A} + q (\mathbf{v} \times \nabla \times \mathbf{A})$$
(2)

The left-hand side of Equation 1 becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \mathbf{v}} \right) = \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{p} + q\mathbf{A}) = \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} + q(\mathbf{v} \cdot \nabla)\mathbf{A} + q\frac{\partial \mathbf{A}}{\partial t}.$$
(3)

Equating Equation 2 and Equation 3 and canceling like terms yields

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = q(\mathbf{v} \times \mathbf{B}) \tag{4}$$

for a stationary magnetic field. Here, the magnetic flux density has been introduced as

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{5}$$

When the particle velocity is small compared to the speed of light Equation 4 yields the classical equation of motion for a charged particle in a stationary, uniform magnetic field

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{v}) = q(\mathbf{v} \times \mathbf{B})$$

# Results and Discussion

The model is solved in COMSOL using the Lagrangian, Hamiltonian, and Newtonian formulations. The Larmor radius is compared to the analytic solution and given in Table 1. All formulations agree with the analytic expression to within 0.05 %. The two coupled first-order differential equations give a slightly different result from a single second-order differential equation for each coordinate.

TABLE I: TABLE COMPARING THE LARMOR RADIUS FOR THE DIFFERENT FORMULATIONS.

	ANALYTIC	LAGRANGIAN	HAMILTONIAN	NEWTONIAN, 2ND ORDER	NEWTONIAN, IST ORDER
Larmor radius (µm)	414.57	414.55	414.55	414.55	414.57



The particle trajectories for the three different formulations are plotted below.

Figure 1: Plot of the ion trajectory for the Lagrangian formulation.



Figure 2: Plot of the ion trajectory for the Hamiltonian formulation.



Figure 3: Plot of the particle trajectory for the Newtonian formulation.



Figure 4: Plot of the particle trajectory for the first-order Newtonian formulation.

Reference

1. L.D. Landau and E.M. Lifshitz, The Classical Theory of Fields, 4th ed., Elsevier, 2005.

# **Application Library path:** Particle\_Tracing\_Module/ Charged Particle Tracing/ion cyclotron motion

# Modeling Instructions

From the File menu, choose New.

# NEW

In the New window, click 🙆 Model Wizard.

# MODEL WIZARD

- I In the Model Wizard window, click 间 3D.
- 2 In the Select Physics tree, select Mathematics>Mathematical Particle Tracing (pt).
- 3 Click Add.
- 4 Click  $\bigcirc$  Study.
- 5 In the Select Study tree, select General Studies>Time Dependent.
- 6 Click **M** Done.

#### GEOMETRY I

Cylinder I (cyl1)

- I In the Geometry toolbar, click 问 Cylinder.
- 2 In the Settings window for Cylinder, locate the Size and Shape section.
- 3 In the Radius text field, type 2e-3.
- 4 In the Height text field, type 2e-3.
- 5 In the Geometry toolbar, click 🟢 Build All.

## GLOBAL DEFINITIONS

Define parameters for the particle mass, magnetic flux density, initial particle velocity, and Larmor radius. The Larmor radius is only used during results processing.

## Parameters 1

- I In the Model Builder window, under Global Definitions click Parameters I.
- 2 In the Settings window for Parameters, locate the Parameters section.

3	In the	table,	enter	the	folle	owing	settings:
		,				<i>D</i>	

Name	Expression	Value	Description
mp	0.04[kg/mol]/ N_A_const	6.6422E-26 kg	Ion mass
В	2[T]	2 T	Magnetic flux density
v0	2E3[m/s]	2000 m/s	Particle velocity, perpendicular to the magnetic field
rL	mp*v0/(e_const*B)	4.1457E-4 m	Larmor radius

# DEFINITIONS

Define an analytic expression for the magnetic vector potential, which results in a uniform magnetic field in the z direction.

Variables I

- I In the Model Builder window, expand the Component I (compl)>Definitions node.
- 2 Right-click Definitions and choose Local Variables.
- 3 In the Settings window for Variables, locate the Variables section.
- **4** In the table, enter the following settings:

Name	Expression	Unit	Description
Ax	1[Wb/m]*y[1/m]	Wb/m	Magnetic vector potential, x-component
Ау	-1[Wb/m]*x[1/m]	Wb/m	Magnetic vector potential, y-component
Az	O[Wb/m]	Wb/m	Magnetic vector potential, z-component
Bx	d(Az,y) - d(Ay,z)	Т	Magnetic flux density, x-component
Ву	d(Ax,z) - d(Az,x)	Т	Magnetic flux density, y-component
Bz	d(Ay,x) - d(Ax,y)	Т	Magnetic flux density, z-component

#### MATHEMATICAL PARTICLE TRACING (PT)

Release a single particle at the origin with an initial velocity in the x direction so that the Lorentz force is nonzero. Also add a small initial velocity in the z direction so that you can clearly see the particle trajectory after solving.

## Release from Grid I

- I In the Model Builder window, under Component I (comp1) right-click Mathematical Particle Tracing (pt) and choose Release from Grid.
- 2 In the Settings window for Release from Grid, locate the Initial Velocity section.
- **3** Specify the  $\mathbf{v}_0$  vector as

v0	x
0	у
1e2	z

The first formulation you will use is **Lagrangian**. The Lagrangian for a particle in a magnetic field is the sum of the particle kinetic energy, which is here defined as pt.Ep, and the dot product of the particle velocity and the magnetic potential, multiplied by the particle charge.

- 4 In the Model Builder window, click Mathematical Particle Tracing (pt).
- **5** In the Settings window for Mathematical Particle Tracing, locate the Particle Release and Propagation section.
- 6 From the Formulation list, choose Lagrangian.

#### Particle Properties 1

- I In the Model Builder window, click Particle Properties I.
- 2 In the Settings window for Particle Properties, locate the Particle Mass section.
- **3** In the  $m_{\rm p}$  text field, type mp.
- 4 Locate the Lagrangian section. In the *L* text field, type pt.Ep+e\_const\*(pt.vx\*Ax+ pt.vy\*Ay+pt.vz\*Az).

# MESH I

Use a coarse mesh. The field is entered using an analytic expression, so the accuracy of the solution is independent of the mesh element size.

- I In the Model Builder window, under Component I (compl) click Mesh I.
- 2 In the Settings window for Mesh, locate the Physics-Controlled Mesh section.
- 3 From the Element size list, choose Extra coarse.

# 4 Click 📗 Build All.

## STUDY I

Step 1: Time Dependent

- I In the Model Builder window, under Study I click Step I: Time Dependent.
- 2 In the Settings window for Time Dependent, locate the Study Settings section.
- 3 In the **Output times** text field, type range(0,5.0e-8,2.0e-5).

For all of the studies in this example, a tight user-defined tolerance will be used to ensure particle kinetic energy is conserved.

- 4 From the Tolerance list, choose User controlled.
- 5 In the **Relative tolerance** text field, type 1.0E-6.
- 6 In the Model Builder window, click Study I.
- 7 In the Settings window for Study, type Lagrangian Study in the Label text field.
- 8 In the Home toolbar, click **= Compute**.

# RESULTS

#### Lagrangian Results

In order to be able to see the radius of the particle orbit, plot the *y*-component of the particle location as a color expression.

- I In the **Settings** window for **3D Plot Group**, type Lagrangian Results in the **Label** text field.
- 2 Locate the Plot Settings section. Clear the Plot dataset edges check box.

#### Particle Trajectories 1

Render the particle trajectory as a ribbon. The default ribbon orientation is in the direction of the unit binormal, or the direction out of the plane tangent to the curved trajectory.

- I In the Model Builder window, expand the Lagrangian Results node, then click Particle Trajectories I.
- 2 In the Settings window for Particle Trajectories, locate the Coloring and Style section.
- 3 Find the Line style subsection. From the Type list, choose Ribbon.
- 4 Select the Width scale factor check box.
- 5 In the associated text field, type 4E-5.
- 6 Find the Point style subsection. From the Type list, choose None.

#### Color Expression 1

- I In the Model Builder window, expand the Particle Trajectories I node, then click Color Expression I.
- 2 In the Settings window for Color Expression, locate the Expression section.
- **3** In the **Expression** text field, type qy/2.
- 4 Locate the Coloring and Style section. From the Color table list, choose Inferno.
- 5 In the Lagrangian Results toolbar, click **I** Plot.
- 6 Click the ↓ Zoom Extents button in the Graphics toolbar. The plot should look like Figure 1.

#### Particle Evaluation 1

- I In the Results toolbar, click <sup>8,85</sup><sub>e-12</sub> More Derived Values and choose Other> Particle Evaluation.
- 2 In the Settings window for Particle Evaluation, locate the Data section.
- 3 From the Time selection list, choose First.
- 4 Locate the Expression section. In the Expression text field, type rL.
- 5 Click the arrow next to the Evaluate button and click New Table.

#### Particle Evaluation 2

- I In the Results toolbar, click <sup>8,85</sup><sub>e-12</sub> More Derived Values and choose Other> Particle Evaluation.
- 2 In the Settings window for Particle Evaluation, locate the Data section.
- **3** From the **Time selection** list, choose **First**.
- 4 Locate the Expression section. In the Expression text field, type timemax(0,2E-5,qy)/ 2.
- 5 Click the arrow next to the Evaluate button and select Table I Particle Evaluation I (rL).

Now switch formulation from Lagrangian to Hamiltonian. When you do this, the particle momentum components are added as additional degrees of freedom. The momentum has three components:  $p_x$ ,  $p_y$ , and  $p_z$ . This results in a doubling of the number of degrees of freedom in the model.

#### MATHEMATICAL PARTICLE TRACING (PT)

- I In the Model Builder window, under Component I (comp1) click Mathematical Particle Tracing (pt).
- 2 In the Settings window for Mathematical Particle Tracing, locate the Particle Release and Propagation section.

3 From the Formulation list, choose Hamiltonian.

## Particle Properties 1

- I In the Model Builder window, under Component I (compl)> Mathematical Particle Tracing (pt) click Particle Properties I.
- 2 In the Settings window for Particle Properties, locate the Hamiltonian section.
- 3 In the H text field, type ((px-e\_const\*Ax)^2+(py-e\_const\*Ay)^2+(pz-e\_const\* Az)^2)/(2\*pt.mp).

## ADD STUDY

- I In the Home toolbar, click  $\sim$  Add Study to open the Add Study window.
- 2 Go to the Add Study window.
- 3 Find the Studies subsection. In the Select Study tree, select General Studies> Time Dependent.
- 4 Click Add Study in the window toolbar.
- 5 In the Home toolbar, click Add Study to close the Add Study window.

# STUDY 2

Step 1: Time Dependent

- I In the Settings window for Time Dependent, locate the Study Settings section.
- 2 In the **Output times** text field, type range(0,5.0e-8,2.0e-5).
- 3 From the Tolerance list, choose User controlled.
- 4 In the Relative tolerance text field, type 1.0E-6.
- 5 In the Model Builder window, click Study 2.
- 6 In the Settings window for Study, type Hamiltonian Study in the Label text field.
- 7 In the **Home** toolbar, click **= Compute**.

## RESULTS

Hamiltonian Results

- I In the Settings window for 3D Plot Group, type Hamiltonian Results in the Label text field.
- 2 Locate the Plot Settings section. Clear the Plot dataset edges check box.

#### Particle Trajectories 1

I In the Model Builder window, expand the Hamiltonian Results node, then click Particle Trajectories I.

- 2 In the Settings window for Particle Trajectories, locate the Coloring and Style section.
- 3 Find the Line style subsection. From the Type list, choose Ribbon.
- 4 Select the Width scale factor check box.
- 5 In the associated text field, type 4E-5.
- 6 Find the Point style subsection. From the Type list, choose None.

#### Color Expression 1

- I In the Model Builder window, expand the Particle Trajectories I node, then click Color Expression I.
- 2 In the Settings window for Color Expression, locate the Expression section.
- **3** In the **Expression** text field, type qy/2.
- 4 Locate the Coloring and Style section. From the Color table list, choose Inferno.
- 5 In the Hamiltonian Results toolbar, click 💽 Plot.
- 6 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 2.

#### Particle Evaluation 3

- I In the Model Builder window, expand the Results>Datasets node.
- 2 Right-click Results>Derived Values and choose More Derived Values>Particle Evaluation.
- 3 In the Settings window for Particle Evaluation, locate the Data section.
- 4 From the **Dataset** list, choose **Particle 2**.
- 5 From the Time selection list, choose First.
- 6 Locate the Expression section. In the Expression text field, type timemax(0,2E-5,qy)/
   2.
- 7 Click the arrow next to the Evaluate button and select Table I Particle Evaluation I (rL).

Switch to the Newtonian formulation and add the Lorentz force manually.

## MATHEMATICAL PARTICLE TRACING (PT)

- I In the Model Builder window, under Component I (compl) click Mathematical Particle Tracing (pt).
- 2 In the Settings window for Mathematical Particle Tracing, locate the Particle Release and Propagation section.
- **3** From the Formulation list, choose Newtonian.

Force 1

- I In the Physics toolbar, click 🔚 Domains and choose Force.
- **2** Select Domain 1 only.
- 3 In the Settings window for Force, locate the Force section.
- **4** Specify the **F** vector as

e\_const\*(Bz\*pt.vy-By\*pt.vz) x e\_const\*(-Bz\*pt.vx+Bx\*pt.vz) y e\_const\*(By\*pt.vx-Bx\*pt.vy) z

## ADD STUDY

- I In the Home toolbar, click 2 Add Study to open the Add Study window.
- 2 Go to the Add Study window.
- 3 Find the Studies subsection. In the Select Study tree, select General Studies> Time Dependent.
- 4 Click Add Study in the window toolbar.
- 5 In the Home toolbar, click 2 Add Study to close the Add Study window.

# STUDY 3

Step 1: Time Dependent

- I In the Settings window for Time Dependent, locate the Study Settings section.
- 2 In the **Output times** text field, type range(0,5.0e-8,2.0e-5).
- 3 From the Tolerance list, choose User controlled.
- 4 In the Relative tolerance text field, type 1.0E-6.
- 5 In the Model Builder window, click Study 3.
- 6 In the Settings window for Study, type Newtonian Study in the Label text field.
- **7** In the **Home** toolbar, click **= Compute**.

# RESULTS

Newtonian Results

- I In the Settings window for 3D Plot Group, type Newtonian Results in the Label text field.
- 2 Locate the Plot Settings section. Clear the Plot dataset edges check box.

#### Particle Trajectories 1

- I In the Model Builder window, expand the Newtonian Results node, then click Particle Trajectories I.
- 2 In the Settings window for Particle Trajectories, locate the Coloring and Style section.
- 3 Find the Line style subsection. From the Type list, choose Ribbon.
- 4 Select the Width scale factor check box.
- 5 In the associated text field, type 4E-5.
- 6 Find the Point style subsection. From the Type list, choose None.

#### Color Expression 1

- I In the Model Builder window, expand the Particle Trajectories I node, then click Color Expression I.
- 2 In the Settings window for Color Expression, locate the Expression section.
- **3** In the **Expression** text field, type qy/2.
- 4 Locate the Coloring and Style section. From the Color table list, choose Inferno.
- 5 In the Newtonian Results toolbar, click 💿 Plot.
- 6 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 3.

#### Particle Evaluation 4

- I In the Results toolbar, click <sup>8,85</sup><sub>e-12</sub> More Derived Values and choose Other> Particle Evaluation.
- 2 In the Settings window for Particle Evaluation, locate the Data section.
- **3** From the **Dataset** list, choose **Particle 3**.
- 4 From the Time selection list, choose First.
- 5 Locate the Expression section. In the Expression text field, type timemax(0,2E-5,qy)/ 2.
- 6 Click the arrow next to the Evaluate button and select Table I Particle Evaluation I (rL).

Finally, switch to the Newtonian, first order formulation.

#### MATHEMATICAL PARTICLE TRACING (PT)

- I In the Model Builder window, under Component I (compl) click Mathematical Particle Tracing (pt).
- 2 In the Settings window for Mathematical Particle Tracing, locate the Particle Release and Propagation section.

3 From the Formulation list, choose Newtonian, first order.

## ADD STUDY

- I In the Home toolbar, click  $\sim 2$  Add Study to open the Add Study window.
- 2 Go to the Add Study window.
- 3 Find the Studies subsection. In the Select Study tree, select General Studies> Time Dependent.
- 4 Click Add Study in the window toolbar.
- 5 In the Home toolbar, click 2 Add Study to close the Add Study window.

## STUDY 4

#### Step 1: Time Dependent

- I In the Settings window for Time Dependent, locate the Study Settings section.
- 2 In the **Output times** text field, type range(0,5.0e-8,2.0e-5).
- 3 From the Tolerance list, choose User controlled.
- 4 In the **Relative tolerance** text field, type 1.0E-6.
- 5 In the Model Builder window, click Study 4.
- 6 In the Settings window for Study, type Newtonian, First Order Study in the Label text field.
- 7 In the **Home** toolbar, click **= Compute**.

# RESULTS

## Newtonian, First Order Results

- I In the Settings window for 3D Plot Group, type Newtonian, First Order Results in the Label text field.
- 2 Locate the Plot Settings section. Clear the Plot dataset edges check box.

## Particle Trajectories I

- I In the Model Builder window, expand the Newtonian, First Order Results node, then click Particle Trajectories I.
- 2 In the Settings window for Particle Trajectories, locate the Coloring and Style section.
- 3 Find the Line style subsection. From the Type list, choose Ribbon.
- 4 Select the Width scale factor check box.
- 5 In the associated text field, type 4E-5.

6 Find the Point style subsection. From the Type list, choose None.

Color Expression 1

- I In the Model Builder window, expand the Particle Trajectories I node, then click Color Expression I.
- 2 In the Settings window for Color Expression, locate the Expression section.
- **3** In the **Expression** text field, type qy/2.
- 4 Locate the Coloring and Style section. From the Color table list, choose Inferno.
- 5 In the Newtonian, First Order Results toolbar, click 🗿 Plot.
- 6 Click the **Go to Default View** button in the **Graphics** toolbar. The plot should look like Figure 4.

Particle Evaluation 5

- I In the Results toolbar, click <sup>8,85</sup><sub>e-12</sub> More Derived Values and choose Other> Particle Evaluation.
- 2 In the Settings window for Particle Evaluation, locate the Data section.
- 3 From the **Dataset** list, choose **Particle 4**.
- 4 From the Time selection list, choose First.
- 5 Locate the Expression section. In the Expression text field, type timemax(0,2E-5,qy)/ 2.
- 6 Click the arrow next to the Evaluate button and select Table I Particle Evaluation I (rL).

The Larmor radius shows good agreement with the analytic expression for all four formulations.