Created in COMSOL Multiphysics 6.0

Nonlinear Propagation of a Cylindrical Wave — Verification Model

Introduction

The linear or small acoustic perturbation theory is in most of the cases sufficient to describe the acoustic phenomena taking place in a particular industrial application. However, when the intensity of sound reaches higher levels, for example, in the majority of the medical ultrasound applications, the small perturbation theory becomes inadequate. In this case, one speaks of the propagation of finite amplitude sound waves. This approach takes into consideration the nonlinear effects that are due to the energy transfer from lower to higher harmonics. The local effective speed of sound becomes larger in the regions of higher sound pressure, which results in distortion of the wave profile and in the end leads to shocks.

The nonlinear effects can be put into two categories: local and cumulative effects. The former usually become negligible once the propagation distance becomes much greater than a wavelength; see Ref. 1. Thus the cumulative effects dominate the local ones under the assumption of progressive waves.

It is obvious that the superposition principle is in general not applicable while modeling nonlinear phenomena. Therefore, a transient analysis is necessary to account for the cumulative distortion along with the wave propagation.

This example shows how to model nonlinear propagation of finite-amplitude acoustic waves in fluids using the *Nonlinear Pressure Acoustics, Time Explicit* physics interface of the Acoustics Module. The interfaces solves the system of nonlinear acoustic equations in the form of a hyperbolic conservation law, see Ref. 2, using the discontinuous Galerkin finite element method (dG-FEM) and explicit time integration scheme. The computed numerical solution for the nonlinear propagation of a cylindrical wave is compared to the analytical solution available before the shock formation.

Model Definition

Consider a finite amplitude acoustic wave propagating in a lossless media in the absence of volume sources. The system of governing equations implemented in the *Nonlinear Pressure Acoustics, Time Explicit* interface, written in the dimensionless form, reads

$$\begin{aligned} \frac{\partial p}{\partial t} + \nabla \cdot ((1+\beta p)\mathbf{u}) &= 0 \\ \frac{\partial \mathbf{u}}{\partial t} + \nabla p &= 0 \end{aligned} \qquad (1)$$

where p is the acoustic pressure, $\mathbf{u}$ is the acoustic particle velocity, and β is the coefficient of nonlinearity. The dimensionless form means that the time, distance, and velocity are scaled to the period, wavelength, and speed of sound, respectively. It is clear that Equation 1 has the form of a hyperbolic conservation law Ref. 3.

Let a cylindrical wave be induced by an acoustic pressure signal $p(t) = P_0\sin(2\pi t)$ prescribed on a circle of the radius r_0. Because of the circular symmetry of the source, the computational domain may be reduced to a circular sector with an arbitrary central angle. In this model, the sector has the angle of 45° as shown in Figure 1.

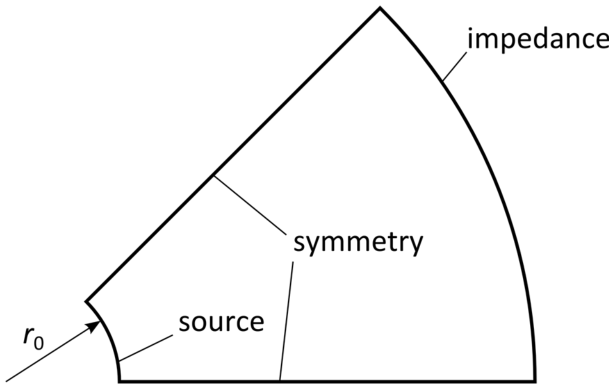

Figure 1: Model geometry.

An impedance boundary condition is imposed on the outer boundary to suppress undesirable reflections.

The described problem has an analytical solution (see Ref. 1). For the given excitation the dimensionless form of the analytical solution reads

$$p_a(r, t) = P_0\sqrt{\frac{r_0}{r}}\sin(2\pi(\tau + \beta z p_a)), \tag{2}$$

where $\tau = t - (r - r_0)$ is the retarded time and $z = 2(\sqrt{rr_0} - r_0)$. Equation 2 is valid for the radii $r \le r_{sh}$, where $r_{sh} = r_0(1 + 1/(4\pi\beta P_0 r_0))^2$ is the shock formation distance.

Results and Discussion

The evolution of the wave traveling from the source is illustrated in Figure 2. The nonlinear behavior becomes more distinct as the distance from the source increases. These are results of the cumulative nonlinear effects. The distortion of the waveform increases with the distance, in the end leading to the formation of shocks, which can be seen as a very sharp transition from the positive (red) to the negative (blue) acoustic pressure closer to the outer boundary.

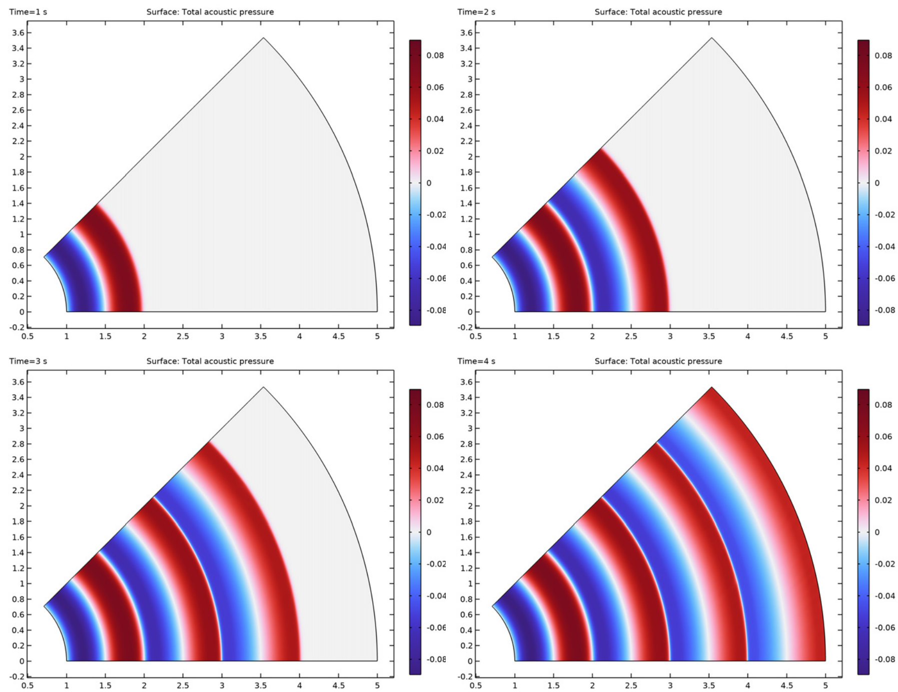

Figure 2: Nonlinear propagation of the cylindrical wave at the times t = 1, 2, 3, and 4.

The formation of shocks is seen in Figure 3. Initially, the waveform distortion is caused by the dependence of local propagation speed on the acoustic pressure. The peaks of the wave travel faster than the troughs, and the waveform takes the shock wave structure after the shock formation (the green vertical line in Figure 3).

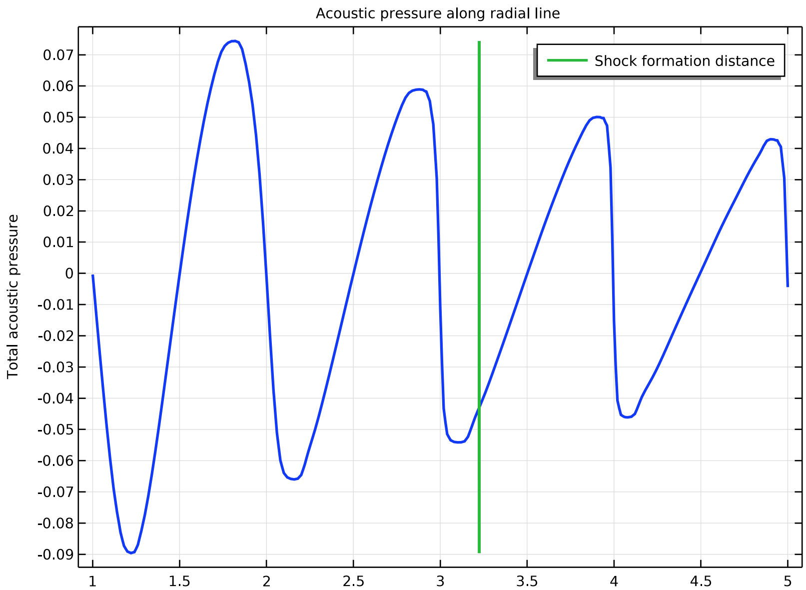

Figure 3: Acoustic pressure along the radial line at the end of the computation.

The distortion of the waveform results in the generation of higher harmonic components. The further the wave travels, the more energy is transferred to the higher harmonic components from the fundamental frequency of the harmonic source signal. This effect is demonstrated in Figure 4 which shows the frequency spectra of the acoustic pressure on the inner (source) and the outer (impedance) boundaries. It is seen that the signal at the source boundary has no frequency components other than the fundamental one. On the other hand, when the wave reaches the outer boundary, the contribution of the higher-order harmonics becomes distinct.

The model solution is compared to the analytical solution obtained from solving nonlinear Equation 2. The results are depicted in Figure 5. There is a good match between the numerical and the analytical solution in both amplitude and phase.

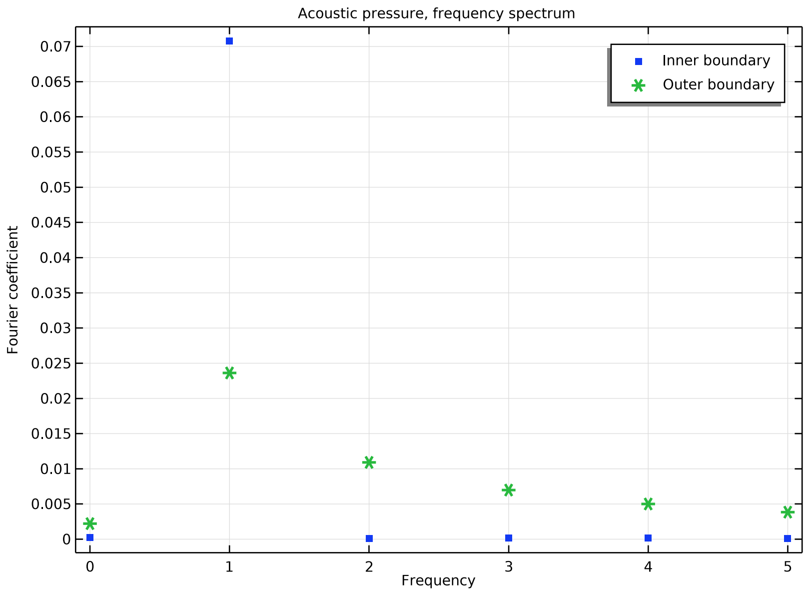

Figure 4: Frequency spectrum of the acoustic pressure signal on the inner and the outer boundaries.

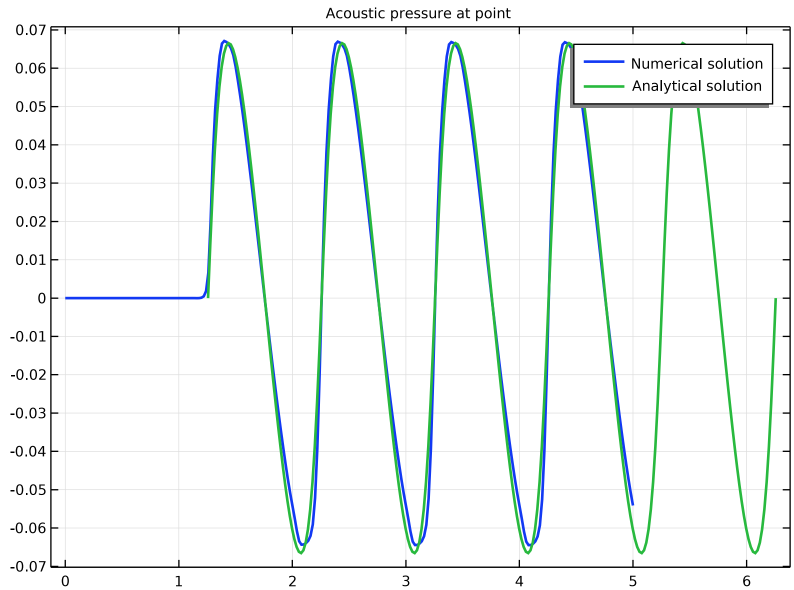

Figure 5: Comparison of model solution (blue) with analytical solution (green) at $r = 0.7r_{\text{sh}}$.

Notes About the COMSOL Implementation

SHOCK LIMITER AND DISCRETIZATION

In this model, the outer radius of the computational domain is chosen to be larger than the shock formation distance r_{sh} and therefore the traveling wave will endure shock discontinuities at the distances $r \geq r_{\text{sh}}$. The treatment of solution discontinuities requires special techniques. One of them is the **WENO Limiter** (Weighted Essentially Non-Oscillatory) available in the *Nonlinear Pressure Acoustic, Time Explicit* physics interface. The use of the **TVB Troubled cell indicator** makes it possible to identify the cells where WENO limiting is needed.

The WENO Limiter does not support discretization orders larger than one. Thus the default **Quartic** discretization has to be changed to **Linear**.

MESH AND TIME EXPLICIT SOLVER

Solving wave propagation problems in the time domain has some requirements on both spatial and temporal resolution of the wave pattern. The mesh has to be fine enough to resolve the frequency content of the signal, that is, the specified number of the higher-order harmonics, N. For the linear discretization, the proper accuracy is achieved when the maximum mesh element size does not exceed 1/10 of the minimum wavelength. That is, $h_{\max} \leq 1/(10N)$.

The *Nonlinear Pressure Acoustic, Time Explicit* physics is based on dG-FEM and uses an explicit time integration schemes. The time step is supposed to obey the CFL condition to ensure the stability of the time integration method. That is, $\Delta t \leq h_{\min}/c_{\max}$, where $h_{\min}$ is the minimum mesh element size and $c_{\max}$ is the maximum wave propagation speed. The latter locally depends on the acoustic pressure $c_{\max} = c_0(1+\beta\max(|p|)/\rho_0 c_0^2)$.

The computation of discontinuous solutions requires that a Strong Stability Preserving (SSP) Runge–Kutta method be used. The third order SSP Runge–Kutta method is achievable by changing the **Order** of the Runge–Kutta method from the default 4 to 3. Since the local speed of sound is not a constant, it is reasonable to enable the option **Update time step** to adjust the time step for a better resolution of the solution.

References

1. M.F. Hamilton and D.T. Blackstock, eds., *Nonlinear Acoustics*, Academic Press, San Diego, CA, 1998.

2. M.A. Diaz, M.A. Solovchuk, and T.W.H. Sheu, "A conservative numerical scheme for modeling nonlinear acoustic propagation in thermoviscous homegeneous media", *J. Comp. Phys.*, vol. 363, 2018.

3. E.F. Toro, *Riemann Solvers and Numerical Methods for Fluid Dynamics. A Practical Introduction*, 3rd Ed., Springer-Verlag, Berlin, 2009.

Application Library path: `Acoustics_Module/Nonlinear_Acoustics/nonlinear_cylindrical_wave`

Modeling Instructions

From the **File** menu, choose **New**.

NEW

In the **New** window, click **Model Wizard**.

MODEL WIZARD

1 In the **Model Wizard** window, click **2D**.
2 In the **Select Physics** tree, select **Acoustics>Ultrasound>Nonlinear Pressure Acoustics, Time Explicit (nate)**.
3 Click **Add**.
4 Click **Study**.
5 In the **Select Study** tree, select **General Studies>Time Dependent**.
6 Click **Done**.

GLOBAL DEFINITIONS

Parameters 1

1 In the **Model Builder** window, under **Global Definitions** click **Parameters 1**.
2 In the **Settings** window for **Parameters**, locate the **Parameters** section.
3 Click **Load from File**.
4 Browse to the model's Application Libraries folder and double-click the file `nonlinear_cylindrical_wave_parameters.txt`.

Before setting up the physics, change the unit system to be dimensionless.

ROOT

1 In the **Model Builder** window, click the root node.
2 In the root node's **Settings** window, locate the **Unit System** section.
3 From the **Unit system** list, choose **None**.

GEOMETRY 1

Circle 1 (c1)

1 In the **Geometry** toolbar, click **Circle**.
2 In the **Settings** window for **Circle**, locate the **Size and Shape** section.
3 In the **Radius** text field, type `r0`.
4 In the **Sector angle** text field, type `45`.
5 Click **Build Selected**.

Circle 2 (c2)

1 Right-click **Circle 1 (c1)** and choose **Duplicate**.
2 In the **Settings** window for **Circle**, locate the **Size and Shape** section.

3 In the **Radius** text field, type `5*r0`.

4 Click **Build Selected**.

Difference 1 (dif1)

1 In the **Geometry** toolbar, click **Booleans and Partitions** and choose **Difference**.

2 Select the object **c2** only.

3 In the **Settings** window for **Difference**, locate the **Difference** section.

4 Find the **Objects to subtract** subsection. Click to select the **Activate Selection** toggle button.

5 Select the object **c1** only.

6 Click **Build All Objects**.

7 Click the **Zoom Extents** button in the **Graphics** toolbar.

Since the outer radius is larger than the shock formation radius, shocks form as the wave passes r_{sh}. Therefore it is required to turn on the Limiter to resolve the shocks.

1 Click the **Show More Options** button in the **Model Builder** toolbar.

2 In the **Show More Options** dialog box, in the tree, select the check box for the node **Physics>Stabilization**.

3 Click **OK**.

NONLINEAR PRESSURE ACOUSTICS, TIME EXPLICIT (NATE)

1 In the **Model Builder** window, under **Component 1 (comp1)** click **Nonlinear Pressure Acoustics, Time Explicit (nate)**.

2 In the **Settings** window for **Nonlinear Pressure Acoustics, Time Explicit**, click to expand the **Limiter** section.

3 From the **Limiter** list, choose **WENO**.

The WENO Limiter does not support discretization orders larger than one. Thus the default **Quartic** discretization has to be changed to **Linear**.

4 Click to expand the **Discretization** section. From the **Element order** list, choose **Linear**.

Nonlinear Pressure Acoustics, Time Explicit Model 1

1 In the **Model Builder** window, under **Component 1 (comp1)>Nonlinear Pressure Acoustics, Time Explicit (nate)** click **Nonlinear Pressure Acoustics, Time Explicit Model 1**.

2 In the **Settings** window for **Nonlinear Pressure Acoustics, Time Explicit Model**, locate the **Pressure Acoustics Model** section.

3 From the c list, choose **User defined**. In the associated text field, type `1`.

4 From the ρ list, choose **User defined**. In the associated text field, type 1.

5 Locate the **Coefficient of Nonlinearity** section. From the β list, choose **User defined**.

6 In the text field, type beta.

Pressure 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Pressure**.

2 Select Boundary 3 only.

3 In the **Settings** window for **Pressure**, locate the **Pressure** section.

4 In the $p_0(t)$ text field, type P0*sin(2*pi*t).

Impedance 1

1 In the **Physics** toolbar, click **Boundaries** and choose **Impedance**.

2 Select Boundary 4 only.

3 In the **Settings** window for **Impedance**, locate the **Impedance** section.

4 From the **Pressure-particle velocity relation** list, choose **Second order**.

MESH 1

Free Triangular 1

In the **Mesh** toolbar, click **Free Triangular**.

Size

1 In the **Model Builder** window, click **Size**.

2 In the **Settings** window for **Size**, locate the **Element Size** section.

3 Click the **Custom** button.

When the linear discretization is used, the mesh should have at least 10 elements per wavelength to resolve the wave pattern.

4 Locate the **Element Size Parameters** section. In the **Maximum element size** text field, type 1/10/N.

5 Click **Build All**.

STUDY 1 - NUMERICAL

1 In the **Model Builder** window, click **Study 1**.

2 In the **Settings** window for **Study**, type Study 1 - Numerical in the **Label** text field.

Solution 1 (sol1)

1 In the **Study** toolbar, click **Show Default Solver**.

2 In the **Model Builder** window, expand the **Solution 1 (sol1)** node, then click **Time-Explicit Solver 1**.

3 In the **Settings** window for **Time-Explicit Solver**, locate the **General** section.

4 From the **Order** list, choose **3**.

This ensures that the third-order Strong Stability-Preserving (SSP) Runge-Kutta solver will be used, which is required for problems with discontinuous solutions (shocks).

5 From the **Update time step** list, choose **Manual**.

Step 1: Time Dependent

1 In the **Model Builder** window, under **Study 1 - Numerical** click **Step 1: Time Dependent**.

2 In the **Settings** window for **Time Dependent**, locate the **Study Settings** section.

3 In the **Output times** text field, type `range(0, 1/50, 5)`.

4 In the **Study** toolbar, click **Compute**.

RESULTS

Acoustic Pressure (nate)

First, inspect the propagation of the wave by looking at its profile at various times. The results should look like the ones in Figure 2.

Then, plot the acoustic pressure along the radial line to see the formation of shocks.

Acoustic Pressure, Line

1 In the **Home** toolbar, click **Add Plot Group** and choose **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, click to expand the **Title** section.

3 From the **Title type** list, choose **Manual**.

4 In the **Title** text area, type `Acoustic pressure along radial line`.

5 Locate the **Data** section. From the **Time selection** list, choose **Last**.

6 In the **Label** text field, type `Acoustic Pressure, Line`.

Line Graph 1

1 In the **Acoustic Pressure, Line** toolbar, click **Line Graph**.

2 Select Boundary 2 only.

3 In the **Settings** window for **Line Graph**, locate the **x-Axis Data** section.

4 From the **Parameter** list, choose **Expression**.

5 In the **Expression** text field, type `x`.

6 Click to expand the **Coloring and Style** section. In the **Width** text field, type `2`.

Line Graph 2

1 Right-click **Line Graph 1** and choose **Duplicate**.

2 In the **Settings** window for **Line Graph**, locate the **x-Axis Data** section.

3 In the **Expression** text field, type `r_sh`.

4 In the **Acoustic Pressure, Line** toolbar, click **Plot**.

5 Click to expand the **Legends** section. Select the **Show legends** check box.

6 From the **Legends** list, choose **Manual**.

7 In the table, enter the following settings:

Legends
`Shock formation distance`

8 In the **Acoustic Pressure, Line** toolbar, click **Plot**.

Line Graph 1

1 In the **Model Builder** window, click **Line Graph 1**.

2 Click **Plot**.

The result should look like the one in Figure 3.

Acoustic Pressure, Frequency Spectrum

1 In the **Home** toolbar, click **Add Plot Group** and choose **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type `Acoustic Pressure, Frequency Spectrum` in the **Label** text field.

3 Locate the **Data** section. From the **Time selection** list, choose **Interpolated**.

4 In the **Times (s)** text field, type `range(4, 1/50, 5)`.

5 Locate the **Title** section. From the **Title type** list, choose **Manual**.

6 In the **Title** text area, type `Acoustic pressure, frequency spectrum`.

Point Graph 1

1 Right-click **Acoustic Pressure, Frequency Spectrum** and choose **Point Graph**.

2 Select Point 2 only.

3 In the **Settings** window for **Point Graph**, locate the **x-Axis Data** section.

4 From the **Parameter** list, choose **Discrete Fourier transform**.

5 From the **Show** list, choose **Frequency spectrum**.

6 From the **Scale** list, choose **Multiply by sampling period**.

7 Select the **Frequency range** check box.

8 In the **Maximum** text field, type N.

9 Click to expand the **Coloring and Style** section. Find the **Line style** subsection. From the **Line** list, choose **None**.

10 Find the **Line markers** subsection. From the **Marker** list, choose **Point**.

11 From the **Positioning** list, choose **In data points**.

12 Click to expand the **Legends** section. Select the **Show legends** check box.

13 From the **Legends** list, choose **Manual**.

14 In the table, enter the following settings:

Legends
Inner boundary

Point Graph 2

1 Right-click **Point Graph 1** and choose **Duplicate**.

2 In the **Settings** window for **Point Graph**, locate the **Selection** section.

3 Click **Clear Selection**.

4 Select Point 4 only.

5 Locate the **Coloring and Style** section. Find the **Line markers** subsection. From the **Marker** list, choose **Asterisk**.

6 Locate the **Legends** section. In the table, enter the following settings:

Legends
Outer boundary

7 In the **Acoustic Pressure, Frequency Spectrum** toolbar, click **Plot**.

The result should look like the one in Figure 4.

ADD PHYSICS

1 In the **Home** toolbar, click **Add Physics** to open the **Add Physics** window.

2 Go to the **Add Physics** window.

3 In the tree, select **Mathematics>ODE and DAE Interfaces>Global ODEs and DAEs (ge)**.

4 Find the **Physics interfaces in study** subsection. In the table, clear the **Solve** check box for **Study 1 - Numerical**.

5 Click **Add to Component 1** in the window toolbar.

6 In the **Home** toolbar, click **Add Physics** to close the **Add Physics** window.

GLOBAL ODES AND DAES (GE)

Global Equations 1

1 In the **Model Builder** window, under **Component 1 (comp1)>Global ODEs and DAEs (ge)** click **Global Equations 1**.

2 In the **Settings** window for **Global Equations**, locate the **Global Equations** section.

3 In the table, enter the following settings:

Name	f(u,ut,utt,t)	Initial value (u_0)	Initial value (u_t0)	Description
pa	pa - P0* sqrt(r0/ (a* r_sh))* sin(2*pi* (tau + 2* (sqrt(a* r_sh*r0) - r0)* beta*pa))	0	0	Analytical solution

ADD STUDY

1 In the **Home** toolbar, click **Add Study** to open the **Add Study** window.

2 Go to the **Add Study** window.

3 Find the **Physics interfaces in study** subsection. In the table, clear the **Solve** check box for **Nonlinear Pressure Acoustics, Time Explicit (nate)**.

4 Find the **Studies** subsection. In the **Select Study** tree, select **General Studies>Stationary**.

5 Click **Add Study** in the window toolbar.

6 In the **Home** toolbar, click **Add Study** to close the **Add Study** window.

STUDY 2 - ANALYTICAL

1 In the **Model Builder** window, click **Study 2**.

2 In the **Settings** window for **Study**, type `Study 2 - Analytical` in the **Label** text field.

3 Locate the **Study Settings** section. Clear the **Generate default plots** check box.

Parametric Sweep

1 In the **Study** toolbar, click **Parametric Sweep**.

2 In the **Settings** window for **Parametric Sweep**, locate the **Study Settings** section.

3 Click **Add**.

4 In the table, enter the following settings:

Parameter name	Parameter value list	Parameter unit
tau (Retarded time)	range(0, 1/50, 5)	

5 In the **Study** toolbar, click **Compute**.

RESULTS

Cut Point 2D 1

1 In the **Results** toolbar, click **Cut Point 2D**.

2 In the **Settings** window for **Cut Point 2D**, locate the **Point Data** section.

3 In the **x** text field, type a*r_sh.

4 In the **y** text field, type 0.

Acoustic Pressure, Point

1 In the **Results** toolbar, click **1D Plot Group**.

2 In the **Settings** window for **1D Plot Group**, type Acoustic Pressure, Point in the **Label** text field.

3 Locate the **Title** section. From the **Title type** list, choose **Manual**.

4 In the **Title** text area, type Acoustic pressure at point.

5 Locate the **Data** section. From the **Dataset** list, choose **None**.

Point Graph 1

1 In the **Acoustic Pressure, Point** toolbar, click **Point Graph**.

2 In the **Settings** window for **Point Graph**, locate the **Data** section.

3 From the **Dataset** list, choose **Cut Point 2D 1**.

4 Locate the **Coloring and Style** section. In the **Width** text field, type 2.

5 Locate the **Legends** section. Select the **Show legends** check box.

6 From the **Legends** list, choose **Manual**.

7 In the table, enter the following settings:

Legends
Numerical solution

Acoustic Pressure, Point

In the **Model Builder** window, click **Acoustic Pressure, Point**.

Global 1

1. In the **Acoustic Pressure, Point** toolbar, click **Global**.
2. In the **Settings** window for **Global**, locate the **Data** section.
3. From the **Dataset** list, choose **Study 2 - Analytical/Solution 2 (sol2)**.
4. Locate the **x-Axis Data** section. From the **Parameter** list, choose **Expression**.

   Make the retarded time parameter match the actual time by adding the information about the signal traveling history from the source to the point ar_{sh}.
5. In the **Expression** text field, type `tau + (a*r_sh - r0)`.
6. Click to expand the **Coloring and Style** section. In the **Width** text field, type `2`.
7. In the **Acoustic Pressure, Point** toolbar, click **Plot**.

   The result should look like the one in Figure 5.